

Odomutants and flexibility results for quantitative orbit equivalence

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April 2, 2025

Abstract

We introduce new systems that we call odomutants, built by distorting the orbits of an odometer. We use these transformations for flexibility results in quantitative orbit equivalence.

It follows from the work of Kerr and Li that if the cocycles of an orbit equivalence are log-integrable, the entropy is preserved. Although entropy is also an invariant of even Kakutani equivalence, we prove that this relation and $L^{<1/2}$ orbit equivalence are not the same, using a non-loosely Bernoulli system of Feldman which is an odomutant.

We also show that Kerr and Li's result on preservation of entropy is optimal, namely we find odomutants of all positive entropies orbit equivalent to an odometer, with almost log-integrable cocycles. We actually build a strong orbit equivalence between uniquely ergodic Cantor minimal homeomorphisms, so our result is a refinement of a famous theorem of Boyle and Handelman.

We finally prove that Belinskaya's theorem is optimal for all the odometers, namely for every odometer, we find a odomutant which is almost-integrably orbit equivalent to it but not flip-conjugate. This yields an extension of a theorem by Carderi, Joseph, Le Maître and Tessera.

Keywords: Quantitative orbit equivalence, odometers, entropy, Kakutani equivalence.

MSC-classification: Primary 37A35; Secondary 37A20.

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1 Introduction

Two ergodic probability measure-preserving bijections S and T on a standard atomless probability space (X, \mathcal{A}, μ) , are *orbit equivalent* if S and some system $\Psi^{-1}T\Psi$ conjugate to T have the same orbits up to measure zero. The isomorphism Ψ is called an *orbit equivalence* between T and S .

A stunning theorem of Dye [Dye59] states that all ergodic measure-preserving bijections of a standard probability space are orbit equivalent. To get a more interesting theory, *quantitative orbit equivalence* proposes to add quantitative restrictions on the *cocycles* associated to orbit equivalence Ψ . These are integer-valued functions c_S and c_T defined by

$$Sx = \Psi^{-1}T^{c_S(x)}\Psi(x) \text{ and } Tx = \Psi S^{c_T(x)}\Psi^{-1}(x),$$

they are well-defined in the ergodic case. In this paper, we consider two quantitative forms of orbit equivalence: *Shannon orbit equivalence* and *φ -integrably orbit equivalence*, for maps $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Shannon orbit equivalence requires that there exists an orbit equivalence whose cocycles are Shannon, meaning that the partitions associated to c_S and c_T are both of finite entropy. For φ -integrable orbit equivalence, we ask that both integrals

$$\int_X \varphi(|c_S(x)|)d\mu(x) \text{ and } \int_X \varphi(|c_T(x)|)d\mu(x)$$

are finite.

In this paper, when $\varphi(x) = x^p$, we are asking that both cocycles c_S and c_T are in L^p , and thus call it an L^p orbit equivalence. Also when c_S and c_T are in L^q for every $q < p$, we say that we have an $L^{<p}$ orbit equivalence. The notion of L^p orbit equivalence can be traced back to the work of Bader, Furman and Sauer [BFS13] in the more general context of measure equivalence, while Shannon orbit equivalence was defined by Kerr and Li. Finally, φ -integrable orbit equivalence was first defined and studied by [DKLMT22].

The main goal is to understand which probability measure-preserving bijections are φ -integrably orbit equivalent or Shannon orbit equivalent. However the construction in the proof of Dye's theorem is not explicit and does not give any quantitative information on the cocycles. Then a more tractable question is the preservation of dynamical properties under these forms of quantitative orbit equivalence. In order to get flexibility results and then partially answer these questions, we introduce in this paper an explicit construction of orbit equivalence between odometers and systems with completely different properties, that we call *odomotants*.

In recent years, odometers have been a central class of systems for explicit constructions, thanks to their combinatorial structure. For example, Kerr and Li [KL24] prove that every odometer is Shannon orbit equivalent to the universal odometer, providing concrete examples of Shannon orbit equivalent systems which are non conjugate. This result was generalized: we show in [Cor25] that many rank-one systems (including the odometers and many irrational rotations) with various spectral and mixing properties are φ -integrably orbit equivalent to the universal odometer, with $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\varphi(x) \underset{x \rightarrow +\infty}{=} o(x^{1/3})$. Finally, in order to show that the main result of [DKLMT22, Theorem 1.1] is optimal in many examples, Delabie, Koivisto, Le Maître and Tessera provide concrete orbit equivalences between group actions¹ built with Følner tilings (see [DKLMT22, Section 6]). It turns out that we get a \mathbb{Z}^k -odometer in the case of the group \mathbb{Z}^k , thus highlighting how useful the combinatorial structures of such systems are.

In our paper, the construction is also based on odometers, it is motivated by a construction by Feldman [Fel76]. The odomotants associated to the same odometer are explicitly built from successive distortions of its orbits, have the same point spectrum (Theorem 3.13) but they can be completely different. They provide flexibility and optimality results: Theorems A, B, C and D that we explain with more details in the following paragraphs.

A theorem of preservation of entropy proved by Kerr and Li. We may wonder whether Shannon or φ -integrable orbit equivalence are trivial or not. Kerr and Li proved that a well-known invariant of conjugacy, the measure-theoretic entropy, is an invariant of Shannon orbit equivalence.

Theorem ([KL24, Theorem A]). *Entropy is preserved under Shannon orbit equivalence.*

A connection between φ -integrable orbit equivalence and Shannon orbit equivalence is given by the following statement which is a consequence of [CJLMT23, Lemma 3.15].

Lemma. *Let $f: X \rightarrow \mathbb{Z}$ be a measurable map. If it is log-integrable, then it is Shannon.*

As a consequence, φ -integrable orbit equivalence implies Shannon orbit equivalence when φ is greater than log and, combined with Kerr and Li's theorem, we get the following result.

Theorem. *Let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a map satisfying $\log t \underset{t \rightarrow +\infty}{=} O(\varphi(t))$. Then entropy is preserved under φ -integrable orbit equivalence.*

¹We do not give any definition in this setting, as the paper is only about probability measure-preserving bijections S , which can be seen as \mathbb{Z} -actions via $(n, x) \in \mathbb{Z} \times X \mapsto S^n x$.

On non-preservation of even Kakutani equivalence. Entropy is also preserved under *even Kakutani equivalence* (see Section 2.4). We may wonder whether there is a connection between this equivalence relation and Shannon orbit equivalence or φ -integrable orbit equivalence for a map $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\log t \underset{t \rightarrow +\infty}{=} O(\varphi(t))$. Note that these quantitative forms of orbit equivalence are not equivalence relations *a priori*. In the result below, $L^{<1/2}$ orbit equivalence means that the cocycles are in L^p for every $p < \frac{1}{2}$.

Theorem A (See Theorem 4.1). *There exists an ergodic probability measure-preserving bijection T which is $L^{<1/2}$ orbit equivalent (in particular Shannon orbit equivalent) to the dyadic odometer but not evenly Kakutani equivalent to it.*

We actually prove that $L^{<1/2}$ orbit equivalence does not preserve loose Bernoullicity², so it does not imply Kakutani equivalence (weaker than even Kakutani equivalence). In [Fel76], Feldman builds a zero-entropy ergodic system which is not loosely Bernoulli. This system, denoted by T , is actually an odomutant built from the dyadic odometer S (this is the first example of odomutant and the starting point of our work). We prove that S and T are $L^{<1/2}$ orbit equivalent and Theorem A follows from the fact that every odometer is loosely Bernoulli.

Remark 1.1. In [Fel76], Feldman did not consider the question of the point spectrum of his non loosely Bernoulli system. As a corollary of Theorem 3.13, we get that it has the same point spectrum as the dyadic odometer.

Question 1.2. Does there exist a sublinear map $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that φ -integrable orbit equivalence implies Kakutani equivalence or even Kakutani equivalence? Such a map would be at least $x \mapsto x^{1/2}$. We may also wonder whether loose Bernoullicity is preserved under φ -integrable orbit equivalence for some sublinear map φ . Note that the case of a linear map φ is straightforward, as a consequence of Belinskaya's theorem.

Optimality result for the preservation of entropy. As stated above, φ -integrable orbit equivalence preserves entropy when the map φ satisfies $\log t \underset{t \rightarrow +\infty}{=} O(\varphi(t))$. Theorem B shows that this result is almost sharp.

Theorem B. *Let (X, μ) be a standard atomless probability space, let α be either a positive real number or $+\infty$, and let $S \in \text{Aut}(X, \mu)$ be an odometer whose associated supernatural number $\prod_{p \in \Pi} p^{k_p}$ satisfies the following property: there exists a prime number p_\star such that $k_{p_\star} = +\infty$. Then there exists a probability measure-preserving transformation $T \in \text{Aut}(X, \mu)$ such that*

1. $h_\mu(T) = \alpha$;
2. *there exists an orbit equivalence between S and T , which is φ_m -integrable for all integers $m \geq 0$,*

where φ_m denotes the map $t \rightarrow \frac{\log t}{\log^{(\circ m)} t}$ and $\log^{(\circ m)}$ the composition $\log \circ \dots \circ \log$ (m times).

The notion of supernatural number associated to an odometer is defined after Definition 2.12, it totally describes its conjugacy class. Examples of odometers S to which this theorem applies are the dyadic odometer, more generally the p -odometer for every prime

²Loosely Bernoulli systems form a class of ergodic systems, which is closed under Kakutani equivalence (see Section 2.4).

number p , or the universal odometer. In our proof, the transformation T is an odomutant associated to S , we now explain how to build such a system.

Theorem B is actually a corollary of Theorem C which is stated in a topological framework. Indeed, to prove this corollary, the main idea was to use topological entropy instead, simpler than measure-theoretic entropy in this context, and connected to it via the variational principle. Moreover, for the topological entropy to be well-defined, we have to consider odomutants that can be extended as homeomorphisms on the Cantor set. We notice that we build a *strong orbit equivalence*, namely an orbit equivalence between homeomorphisms on the Cantor set such that the equality of the orbits holds at every point of the space (and not up to measure zero), and whose associated cocycles each have at most one point of discontinuity.

Theorem C (See Theorem 5.1). *Let α be either a positive real number or $+\infty$. Let S be an odometer whose associated supernatural number $\prod_{p \in \Pi} p^{k_p}$ satisfies the following property: there exists a prime number p_\star such that $k_{p_\star} = +\infty$. Then there exists a Cantor minimal homeomorphism T such that*

1. $h_{\text{top}}(T) = \alpha$;
2. *there exists a strong orbit equivalence between S and T , which is φ_m -integrable for all integers $m \geq 0$,*

where φ_m denotes the map $t \rightarrow \frac{\log t}{\log^{(om)} t}$ and $\log^{(om)}$ the composition $\log \circ \dots \circ \log$ (m times).

In order to create topological entropy, we build an odomutant T from the odometer S in such a way that the dynamics of T describes more words $\{\mathcal{P}(T^i(x))_{0 \leq i \leq n-1} \mid x \in X\}$ than S does, for partitions \mathcal{P} in clopen sets that we will define³. Note that this is more or less the strategy applied by Feldman for the construction of a non loosely Bernoulli system, since loose Bernoullicity property also deals with the words produced by a system. Then Theorem B follows from Theorem C and the variational principle since such a transformation T is necessarily uniquely ergodic (see Proposition 2.20).

For the study of strong orbit equivalence, *Bratteli diagrams* have played a crucial role. Every properly ordered Bratteli diagram provides a Cantor minimal homeomorphism, called a Bratteli-Vershik system. Conversely, Herman, Putnam and Skau proved in [HPS92] that every Cantor minimal homeomorphism is topologically conjugate to a *Bratteli-Vershik system*. Moreover, using this characterization, Giordano, Putnam and Skau completely classified the Cantor minimal homeomorphisms up to strong orbit equivalence, using the *dimension group* which turns out to be a complete invariant (see [GPS95]). We refer the reader to Appendix B for a brief overview.

An earlier version of Theorem B (and more generally Theorem C) stated that there exists an odomutant with positive entropy which is orbit equivalent to an odometer with almost log-integrable cocycles. Thanks to a suggestion of Thierry Giordano, we noticed that odomutants have already appeared in [BH94]. Indeed Boyle and Handelman stated a result similar to Theorem C, without any quantitative information on the cocycles.

Theorem ([BH94, Theorem 2.8 and Section 3]). *Let S be the dyadic odometer. If α is a positive real number or $\alpha = +\infty$, then there exists a Cantor minimal homeomorphism T such that:*

³ $\mathcal{P}(y)$ denotes the atom of the partition \mathcal{P} which contains $y \in X$.

1. $h_{\text{top}}(T) = \alpha$;
2. S and T are strongly orbit equivalent.

Their proof exactly consists in building a Bratteli diagram of an odomutant associated to the dyadic odometer. We thus manage to give a similar statement but with quantitative information on the cocycles (Theorem C). The case of the finite entropy is an improvement of our earlier proof, and the case of the infinite entropy is a translation of Boyle and Handel's proof in our formalism.

Another crucial point is that the orbit equivalence we build in our paper is explicit, whereas Boyle and Handel use the dimension group and so establish the strong orbit equivalence in a more abstract way. The comparison between Boyle and Handel's construction and our formalism will be detailed in Appendix B.

Optimality of Belinskaya's theorem. Belinskaya's theorem [Bel69] states that if S and T are orbit equivalent and one of the two associated cocycles is integrable, then S and T are flip-conjugate, meaning that S is conjugate to T or T^{-1} . As a consequence, L^1 orbit equivalence is exactly flip-conjugacy. Since integrability exactly means φ -integrability for linear maps φ , it is interesting to study the sublinear case, as was done in [CJLMT23].

Theorem ([CJLMT23, Theorem 1.3]). *Let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a sublinear function⁴. Let S be an ergodic probability measure-preserving transformation and assume that S^n is ergodic for some $n \geq 2$. Then there is another ergodic probability measure-preserving transformation T such that S and T are φ -integrably orbit equivalent but not flip-conjugate.*

The authors asked whether this holds for a system S such that S^n is non-ergodic for all $n \geq 2$. The following statement provides an answer for the odometers which satisfy this property.

Theorem D (See Theorem 6.1). *Let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a sublinear map and S an odometer. There exists a probability measure-preserving transformation T such that S and T are φ -integrably orbit equivalent but not flip-conjugate.*

As in the proofs of Theorems A and C, the counter-example T for Theorem D is again an odomutant associated to S . To ensure that S and T are not flip-conjugate, we notice that an odometer is a factor of its associated odomutants, and we use the property of *coalescence* for the odometers, which states that an extension of an odometer is conjugate to it if and only if every factor map associated to this extension is an isomorphism.

Remark 1.3. Note that a probability measure-preserving transformation S such that S^n is non-ergodic for every $n \geq 2$ factors onto some odometer. It would be interesting to combine the proof of Theorem D with this remark so as to completely remove the assumption that S^n is ergodic for some $n \geq 2$ in [CJLMT23, Theorem 1.3].

Outline of the paper. After a few preliminaries in Section 2, we introduce the notion of odomutants in Section 3, we study its measure-theoretic and topological properties, and the orbit equivalence with their associated odometers. Theorems A, C and D are respectively proven in Sections 4, 5 and 6. Appendix A deals with combinatorial results preparing for the

⁴This means that $\lim_{t \rightarrow +\infty} \frac{\varphi(t)}{t} = 0$.

proof of Theorem C. In Appendix B, we describe odomutants as Bratteli-Vershik systems and compare our proof of Theorem C with the proof of Boyle and Handelman's theorem in [BH94]. Finally Appendix C is devoted to prove the well-known (but left unproved in the literature) equivalence between definitions of loose Bernoullicity in the zero-entropy case.

Acknowledgements. I thank my advisors François Le Maître and Romain Tessera for their support and valuable advice on writing this paper. I also thank Fabien Durand and Samuel Petite for fruitful discussion about Cantor minimal systems. Finally, I am very grateful to Thierry Giordano for enlightening conversations about Boyle and Handelman's works and more generally the notion of strong orbit equivalence.

2 Preliminaries

2.1 Basic definitions in ergodic theory

In a measure-theoretic framework. The author may refer to [KL16] and [VO16] for complete surveys about the notions introduced in this section.

The probability space (X, \mathcal{A}, μ) is assumed to be standard and atomless. Such a space is isomorphic to $([0, 1], \mathcal{B}([0, 1]), \text{Leb})$, i.e. there exists a bimeasurable bijection $\Psi: X \rightarrow [0, 1]$ (defined almost everywhere) such that $\Psi_*\mu = \text{Leb}$, where $\Psi_*\mu$ is defined by $\Psi_*\mu(A) = \mu(\Psi^{-1}(A))$ for every measurable set A . We consider maps $T: X \rightarrow X$ acting on this space and which are bijective, bimeasurable and **probability measure-preserving (p.m.p.)**, meaning that $\mu(T^{-1}(A)) = \mu(A)$ for all measurable sets $A \subset X$, and the set of these transformations is denoted by $\text{Aut}(X, \mathcal{A}, \mu)$, or simply $\text{Aut}(X, \mu)$, two such maps being identified if they coincide on a measurable set of full measure. In this paper, elements of $\text{Aut}(X, \mu)$ are called **transformations** or **(dynamical) systems**.

A measurable set $A \subset X$ is **T -invariant** if $\mu(T^{-1}(A)\Delta A) = 0$, where Δ denotes the symmetric difference. The system $T \in \text{Aut}(X, \mu)$ is **(μ -)ergodic**, or μ is **T -ergodic**, if every T -invariant set is of measure 0 or 1. If T is ergodic, then T is **aperiodic**, i.e. $T^n(x) \neq x$ for almost every $x \in X$ and for every $n \in \mathbb{Z} \setminus \{0\}$, or equivalently the **T -orbit** of x , denoted by $\text{Orb}_T(x) := \{T^n(x) \mid n \in \mathbb{Z}\}$, is infinite for almost every $x \in X$. A transformation T is **uniquely ergodic** on X if it admits a unique T -invariant probability measure μ . In this case, μ is T -ergodic since in full generality the extremal points of the convex set of T -invariant probability measures are exactly the ergodic ones.

Denoting by $L^2(X, \mathcal{A}, \mu)$ the space of complex-valued and square-integrable functions defined on X , a complex number λ is an **eigenvalue** of T if there exists $f \in L^2(X, \mathcal{A}, \mu) \setminus \{0\}$ such that $f \circ T = \lambda f$ almost everywhere (f is then called an **eigenfunction**). An eigenvalue λ is automatically an element of the unit circle $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$. The **point spectrum** of T , denoted by $\text{Sp}(T)$, is then the set of all its eigenvalues. Notice that $\lambda = 1$ is always an eigenvalue since the constant functions are in its eigenspace. Moreover T is ergodic if and only if the constant functions are the only eigenfunctions with eigenvalue one, in other words the eigenspace of $\lambda = 1$ is the line of constant functions (we say that it is a simple eigenvalue). Finally, a system has **discrete spectrum** if the span of all its eigenfunctions is dense in $L^2(X, \mathcal{A}, \mu)$.

All the properties that we have introduced are preserved under conjugacy. Two transformations $T \in \text{Aut}(X, \mu)$ and $S \in \text{Aut}(Y, \nu)$ are **conjugate** if there exists a bimeasurable bijection $\Psi: X \rightarrow Y$ such that $\Psi_*\mu = \nu$ and $\Psi \circ T = S \circ \Psi$ almost everywhere. Some classes of transformations have been classified up to conjugacy, the two examples to keep

in mind are the following. By Ornstein [Orn70], entropy is a total invariant of conjugacy among Bernoulli shifts (entropy will be introduced in Section 2.3). Moreover Halmos and von Neumann [HVN42] prove that two ergodic systems with discrete spectrums are conjugate if and only if they have equal point spectrums. For example, the odometers (introduced in Section 2.5) have discrete spectrum and this theorem enables us to classify them up to conjugacy.

Transformations T and S are said to be **flip-conjugate** if T is conjugate to S or to S^{-1} . Since the point spectrum forms a circle subgroup, the Halmos-von Neumann theorem actually states that the point spectrum is a total invariant of flip-conjugacy in the class of ergodic discrete spectrum systems. Therefore we are able to classify the odometers up to flip-conjugacy.

We say that S is a **factor** of T , or T is an **extension** of S , if there exists a measurable map $\Psi: X \rightarrow Y$ which is onto and such that $\Psi_*\nu = \mu$ and $S \circ \Psi = \Psi \circ T$ almost everywhere. The map Ψ is called a **factor map** from T to S .

In a topological framework. The notions that we have introduced are part of a measure-theoretic setting. On the topological side, a **topological (dynamical) system** is a continuous map $T: X \rightarrow X$ on a topological space X (usually X is assumed to be compact). Two topological systems T and S , respectively on topological spaces X and Y , are **topologically conjugate** if there exists a homeomorphism $\Psi: X \rightarrow Y$ such that $\Psi \circ T = S \circ \Psi$ on X . A topological system is **minimal** if every orbit is dense. In this paper, we will only consider **Cantor minimal homeomorphisms**, namely minimal invertible topological systems on the Cantor set.

In this paper, "systems", "conjugacy", "entropy" will always refer to the measure-theoretic setting. For the topological setting, we will always specify "topological system", "topological conjugacy", "topological entropy".

2.2 Measurable partitions

A set \mathcal{P} of measurable subsets of X is a **measurable partition** of X if:

- for every $P_1, P_2 \in \mathcal{P}$, we have $\mu(P_1 \cap P_2) = 0$;
- the union $\bigcup_{P \in \mathcal{P}} P$ has full measure.

The elements of \mathcal{P} are called the **atoms**. If \mathcal{P} and \mathcal{Q} are measurable partitions of (X, μ) , we say that \mathcal{P} **refines** (or is a refinement of, or is finer than) \mathcal{Q} , denoted by $\mathcal{P} \succcurlyeq \mathcal{Q}$, if every atom of \mathcal{Q} is a union of atoms of \mathcal{P} (up to a null set). More generally, their **joint partition** is

$$\mathcal{P} \vee \mathcal{Q} := \{P \cap Q \mid P \in \mathcal{P}, Q \in \mathcal{Q}\},$$

namely the coarsest partition which refines \mathcal{P} and \mathcal{Q} .

A measurable partition \mathcal{P} defines almost everywhere a map $\mathcal{P}(\cdot): X \rightarrow \mathcal{P}$ where $\mathcal{P}(x)$ is the atom of \mathcal{P} which contains x . Given a measurable map $T: X \rightarrow X$, \mathcal{P} provides **coding maps**

$$[\mathcal{P}]_{i,n}: x \in X \mapsto (\mathcal{P}(T^j x))_{i \leq j \leq n} \in \mathcal{P}^{\{i, \dots, n\}}.$$

In particular, $[\mathcal{P}]_n(x) := [\mathcal{P}]_{0,n-1}(x)$ is the **n -word** of x .

Given atoms P_i, P_{i+1}, \dots, P_n of \mathcal{P} , the equality $[\mathcal{P}]_{i,n}(x) = (P_i, \dots, P_n)$ exactly means that x is an element of $T^{-i}(P_i) \cap T^{-(i+1)}(P_{i+1}) \cap \dots \cap T^{-n}(P_n)$. Therefore the partition which gives the values of $[\mathcal{P}]_{i,n}$ is the following joint partition

$$\mathcal{P}_i^n := \bigvee_{j=i}^n T^{-j}(\mathcal{P})$$

with $T^{-j}(\mathcal{P}) := \{T^{-j}(P) \mid P \in \mathcal{P}\}$, this is a division of the space given by the dynamics of T , over the timeline $\{i, \dots, n\}$ and with respect to \mathcal{P} .

2.3 Measure-theoretic entropy, topological entropy

Here we present two notions of entropy. For more details, the reader may refer to [Dow11] and [KL16].

Measure-theoretic entropy. Entropy, or measure-theoretic entropy, or metric entropy, of a measurable transformation is an invariant of conjugacy. To define it, we first define the entropy of a partition, which then enables us to quantify how much a transformation complexifies the partitions.

Let T be a system on (X, μ) , not necessarily invertible, and \mathcal{P} a finite measurable partition of X . Let us define the **entropy of \mathcal{P}** by

$$H_\mu(\mathcal{P}) := - \sum_{P \in \mathcal{P}} \mu(P) \log \mu(P),$$

where $\mu(P) \log \mu(P) = 0$ if P is a null set. This is a positive real number. The following quantity

$$h_\mu(T, \mathcal{P}) := \lim_{n \rightarrow +\infty} \frac{H_\mu(\mathcal{P}_0^{n-1})}{n}$$

is well-defined, this is the **entropy of T with respect to \mathcal{P}** , and it tells us how quickly the dynamics of T is dividing the space X with the partition \mathcal{P} . Finally, let us define the **entropy of T** by

$$h_\mu(T) := \sup_{\mathcal{P}} h_\mu(T, \mathcal{P}),$$

where the supremum is over all the finite measurable partitions \mathcal{P} of X . This quantity is non-negative and can be infinite.

The following result, due to Kolmogorov and Sinai, enables us to prove the well-known fact that the odometers have zero entropy (see Section 2.5).

Theorem 2.1 ([Dow11, after Definition 4.1.1]). *Let $(\mathcal{P}_k)_{k \geq 0}$ be an increasing sequence of partitions which generates the σ -algebra of X (up to restriction to full-measure sets). Then we have*

$$h_\mu(T, \mathcal{P}_k) \xrightarrow{k \rightarrow +\infty} h_\mu(T).$$

Topological entropy. In the topological setting, topological entropy is an invariant of topological conjugacy and is defined with similar ideas.

The topological space X has to be compact. We define the **joint cover** of two open covers \mathcal{U} and \mathcal{V} by

$$\mathcal{U} \vee \mathcal{V} := \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}.$$

Let T be a topological system on X and \mathcal{U} an open cover of X . Let us define

$$\mathcal{U}_0^{n-1} := \bigvee_{i=0}^{n-1} T^{-i}(\mathcal{U}),$$

where $T^{-i}(\mathcal{U}) := \{T^{-i}(U) \mid U \in \mathcal{U}\}$, and

$$\mathcal{N}(\mathcal{U}) := \min\{|\mathcal{U}'| \mid \mathcal{U}' \text{ is a subcover of } \mathcal{U}\},$$

where $|\mathcal{U}'|$ denotes the cardinality of \mathcal{U}' . The quantity $\mathcal{N}(\mathcal{U})$ is finite since X is compact.

The **topological entropy of T with respect to the open cover \mathcal{U}** is the well-defined limit

$$h_{\text{top}}(T, \mathcal{U}) := \lim_{n \rightarrow +\infty} \frac{\log \mathcal{N}(\mathcal{U}_0^{n-1})}{n},$$

it tells us how quickly the dynamics of T is shrinking the open sets of \mathcal{U} .

Finally, let us define the **topological entropy of T** by

$$h_{\text{top}}(T) := \sup_{\mathcal{U}} h_{\text{top}}(T, \mathcal{U}),$$

where the supremum is over all the open covers \mathcal{U} of X . This quantity is non-negative and can be infinite.

The following result will enables us to build an odomutant with a prescribed topological entropy (see Lemma 5.3). We say that a sequence $(\mathcal{U}_n)_{n \geq 0}$ of open covers generates the topology on X if for every $\varepsilon > 0$, there exists $N \geq 0$ such that for every $n \geq N$, the open sets of \mathcal{U}_n has a diameter less than ε .

Theorem 2.2 ([Dow11, Remark 6.1.7]). *Let T be a topological system on X and $(\mathcal{U}_n)_{n \geq 0}$ a generating sequence of open covers. Then we have*

$$h_{\text{top}}(T) = \lim_{n \rightarrow +\infty} h_{\text{top}}(T, \mathcal{U}_n).$$

Example 2.3. The compact space X that we consider in this paper is of the form

$$X := \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\},$$

with integers q_n greater or equal to 2. It admits open covers which are partitions in clopen sets. If \mathcal{U} is such an open cover, then \mathcal{U}_0^{n-1} denotes both joint of open covers and joint of partitions. We have $\mathcal{N}(\mathcal{U}_0^{n-1}) = |\mathcal{U}_0^{n-1} \setminus \{\emptyset\}|$ and this is exactly the number of words of the form $[\mathcal{U}]_n(x)$, for $x \in X$, where $[\mathcal{U}]_n$ is the coding map associated to the partition \mathcal{U} (see Section 2.2). Therefore, in the proof of Theorem C, a method to create topological entropy consists in building a system T whose number of n -words (with respect to some partition in clopen sets) increases quickly enough as n goes to ∞ .

More precisely, the open covers \mathcal{U} that we will consider are

$$\mathcal{P}(\ell) := \{[i_0, \dots, i_{\ell-1}]_\ell \mid 0 \leq i_0 < q_0, \dots, 0 \leq i_{\ell-1} < q_{\ell-1}\},$$

for $\ell \geq 1$, where $[i_0, \dots, i_{\ell-1}]_\ell$ denotes the ℓ -cylinder

$$\{x = (x_n)_{n \geq 0} \mid x_0 = i_0, \dots, x_{\ell-1} = i_{\ell-1}\}.$$

Note that $(\mathcal{P}(\ell))_{\ell \geq 1}$ is a generating sequence of open covers. In Definition 3.3, we will also consider other partitions $\tilde{\mathcal{P}}(\ell)$, for some reasons explained in the paragraph following this definition.

The variational principle. In Example 2.3, we explain the method that we will apply in this paper to create topological entropy and then prove Theorem C. However we also would like to create measure-theoretic entropy to prove Theorem B. The variational principle enables us to connect these notions.

Theorem 2.4 (Variational principle [Dow11, Theorem 6.8.1]). *Let $T: X \rightarrow X$ be a topological system on a metric compact set X . Then we have*

$$h_{\text{top}}(T) = \sup_{\mu} h_{\mu}(T),$$

where the supremum is over all the T -invariant Borel probability measures μ on X .

As a consequence, if T is uniquely ergodic, then we have

$$h_{\text{top}}(T) = h_{\mu}(T),$$

where μ denotes the only T -invariant Borel probability measure.

2.4 Even Kakutani equivalence, loose Bernoullicity

The notions introduced in this section can be found in [Fel76] and [ORW82].

Let $T \in \text{Aut}(X, \mu)$. Given a measurable set A , the return time $r_A: A \rightarrow \mathbb{N}^* \cup \{\infty\}$ is defined by:

$$\forall x \in A, r_A(x) := \inf \{k \geq 1 \mid T^k x \in A\}.$$

It follows from Poincaré recurrence theorem that, if A has positive measure, then the set $\{k \in \mathbb{N}^* \mid T^k x \in A\}$ is infinite for almost every $x \in A$. In particular, $r_A(x)$ is finite for almost every $x \in A$.

Then we can define a transformation T_A on the set $\{x \in A \mid r_A(x) < \infty\}$, namely on A up to a null set, called the **induced transformation** on A :

$$T_A x := T^{r_A(x)} x.$$

The map T_A is an element of $\text{Aut}(A, \mu_A)$, where $\mu_A := \mu(\cdot)/\mu(A)$ is the conditional probability measure. Its entropy is given by Abramov's formula:

$$h_{\mu_A}(T_A) = \frac{h_{\mu}(T)}{\mu(A)}.$$

Definition 2.5. Let $S \in \text{Aut}(X, \mu)$, $T \in \text{Aut}(Y, \nu)$ be two ergodic transformations.

1. T and S are said to be **Kakutani equivalent** if T_A and S_B are isomorphic for some measurable sets $A \subset X$ and $B \subset Y$.
2. Moreover they are **evenly Kakutani equivalent** if in addition two such measurable sets have the same measure: $\mu(A) = \nu(B)$.

It is well-known that Kakutani equivalence and even Kakutani equivalence are equivalence relations. It follows from Abramov's formula that entropy is preserved under even Kakutani equivalence.

Similarly to Ornstein's theory [Orn70] for the conjugacy problem, Ornstein, Rudolph and Weiss [ORW82] found a class of systems, called loosely Bernoulli system, where Kakutani and even Kakutani equivalences are well understood. These systems were first introduced by Feldman [Fel76].

Definition 2.6 (see [Fel76]).

- The f -metric between words of same length is defined by:

$$f_n((a_i)_{1 \leq i \leq n}, (b_i)_{1 \leq i \leq n}) = 1 - \frac{k}{n}$$

where k is the greatest integer for which we can find equal subsequences $(a_{i_\ell})_{1 \leq \ell \leq k}$ and $(b_{j_\ell})_{1 \leq \ell \leq k}$, with $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$.

- Let $T \in \text{Aut}(X, \mu)$ and \mathcal{P} be a partition of X . The couple (T, \mathcal{P}) , called a process, is **loosely Bernoulli** if for every $\varepsilon > 0$, for every sufficiently large integer N and for each $M > 0$, there exists a collection \mathcal{G} of "good" atoms in \mathcal{P}_{-M}^0 whose union has measure greater than or equal to $1 - \varepsilon$, and so that for each pair A, B of atoms in \mathcal{G} , the following holds: there is a probability measure $n_{A,B}$ on $\mathcal{P}^N \times \mathcal{P}^N$ satisfying

1. $n_{A,B}(\{w\} \times \mathcal{P}^N) = \mu_A(\{[\mathcal{P}]_{1,N}(\cdot) = w\})$ for every $w \in \mathcal{P}^N$;
2. $n_{A,B}(\mathcal{P}^N \times \{w'\}) = \mu_B(\{[\mathcal{P}]_{1,N}(\cdot) = w'\})$ for every $w' \in \mathcal{P}^N$;
3. $n_{A,B}(\{(w, w') \in \mathcal{P}^N \times \mathcal{P}^N \mid f_N(w, w') > \varepsilon\}) < \varepsilon$.

- T is **loosely Bernoulli** if (T, \mathcal{P}) is loosely Bernoulli for all finite partitions \mathcal{P} of X .

Loose Bernoullicity for a process (T, \mathcal{P}) expresses the fact that, conditionally to two pasts A and B , the laws for the future are close, meaning that there exists a good coupling between them, with close words for the f -metric.

Example 2.7. The Bernoulli shift on $\{1, \dots, k\}^{\mathbb{Z}}$ is loosely Bernoulli with respect to the partition $\{[1]_1, \dots, [k]_1\}$. Indeed, conditionally to every past, the law for the N -word is always the uniform distribution on $\{1, \dots, k\}^N$, so it suffices to define $n_{A,B}$ as the uniform distribution on the diagonal of $\mathcal{P}^N \times \mathcal{P}^N$, with the notations of the previous definition. This system is more generally loosely Bernoulli since $\{[1]_1, \dots, [k]_1\}$ is a generating partition⁵.

We will also prove that odometers are loosely Bernoulli (see Proposition 2.15 in the next section), using the following equivalent definition of loose Bernoullicity for zero-entropy systems.

Theorem 2.8. *Let $T \in \text{Aut}(X, \mu)$ and \mathcal{P} be a partition of X and assume that $h_\mu(T, \mathcal{P}) = 0$. Then (T, \mathcal{P}) is loosely Bernoulli if and only if for every $\varepsilon > 0$ and for every sufficiently large integer N , there exists a collection \mathcal{H} of "good" atoms in \mathcal{P}_1^N whose union has measure greater than or equal to $1 - \varepsilon$ and so that we have $f_N(w, w') \leq \varepsilon$ for every $w, w' \in [\mathcal{P}]_{1,N}(\mathcal{H})$.*

This has been stated by Feldman [Fel76, Remark in p. 22] and Ornstein, Rudolph and Weiss [ORW82, after Definition 6.1] for instance. However, to our knowledge, there is no justification of this statement in the literature. This is the reason why we provide a proof in Appendix C.

The choice of the metric is very important. Indeed, with the d -metric:

$$d_n((a_i)_{1 \leq i \leq n}, (b_i)_{1 \leq i \leq n}) = |\{1 \leq i \leq n \mid a_i \neq b_i\}|,$$

⁵To prove that a system is loosely Bernoulli, it is enough to prove it with respect to a generating partition (see [ORW82] and the equivalent notion of finitely fixed process).

also called the Hamming distance, we get the notion of very weakly Bernoulli systems and this is exactly the class considered in Ornstein's theory for the conjugacy problem.

As mentioned above, Kakutani equivalence and even Kakutani equivalence are well understood in the class of loosely Bernoulli systems.

Theorem 2.9 ([ORW82, Theorems 5.1 and 5.2]). *Let $S \in \text{Aut}(X, \mu)$, $T \in \text{Aut}(Y, \nu)$ be two ergodic transformations.*

1. *If S is loosely Bernoulli and is Kakutani equivalent to T , then T is also loosely Bernoulli.*
2. *If S and T are loosely Bernoulli, then they are evenly Kakutani equivalent if and only if they have the same entropy.*

2.5 Odometers

Given integers q_0, q_1, q_2, \dots greater than or equal to 2, let us consider the Cantor space

$$X := \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\},$$

endowed with the infinite product topology and the associated Borel σ -algebra. The **odometer** on X is the adding machine $S: X \rightarrow X$, defined for every $x \in X$ by

$$Sx = \begin{cases} (\underbrace{0, \dots, 0}_{i \text{ times}}, 1 + x_i, x_{i+1}, \dots) & \text{if } i := \min \{j \geq 0 \mid x_j \neq q_j - 1\} \text{ is finite} \\ (0, 0, 0, \dots) & \text{if } x = (q_0 - 1, q_1 - 1, q_2 - 1, \dots) \end{cases}.$$

In other words, S is the addition by $(1, 0, 0, \dots)$ with carry over to the right.

An odometer is more generally a system which is conjugate to S for some choice of integers q_n . In this paper, we only consider this concrete example with the adding machine and we refer to it as "the odometer on $\prod_{n \geq 0} \{0, 1, \dots, q_n - 1\}$ ".

Let us introduce the **cylinders of length k** , or **k -cylinders**,

$$[x_0, \dots, x_{k-1}]_k := \left\{ (y_n)_{n \geq 0} \in \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\} \mid y_0 = x_0, \dots, y_{k-1} = x_{k-1} \right\}.$$

We can define a cylinder with a subset I_j of $\{0, 1, \dots, q_j - 1\}$ instead of x_j . For instance, $[x_0, I_1, x_2]_3$ denotes the set of sequences $(y_n)_{n \geq 0}$ satisfying $y_0 = x_0$, $y_1 \in I_1$ and $y_2 = x_2$. We also use the symbol \bullet when we do not want to fix the value at some coordinate. For instance, $[x_0, \bullet, x_2]_3$ denotes the set of sequences $(y_n)_{n \geq 0}$ satisfying $y_0 = x_0$ and $y_2 = x_2$. By convention, the 0-cylinder is X . For any $n \geq 1$, we also set a partially defined map

$$\zeta_n: X \setminus [\bullet, \dots, \bullet, q_{n-1} - 1]_n \rightarrow X \setminus [\bullet, \dots, \bullet, 0]_n$$

which is the addition by

$$(\underbrace{0, \dots, 0}_{n-1 \text{ times}}, 1, 0, 0, \dots)$$

with carry over to the right, and which coincides with $S^{q_0 \dots q_{n-2}}$ on $X \setminus [\bullet, \dots, \bullet, q_{n-1} - 1]_n$. As illustrated in Figure 1, the cylinders and the maps ζ_n offer a very interesting combinatorial structure with successive nested towers $\mathcal{R}_1, \mathcal{R}_2, \dots$.⁶

⁶This kind of construction that we see in Figure 1 is called a cutting-and-stacking construction.

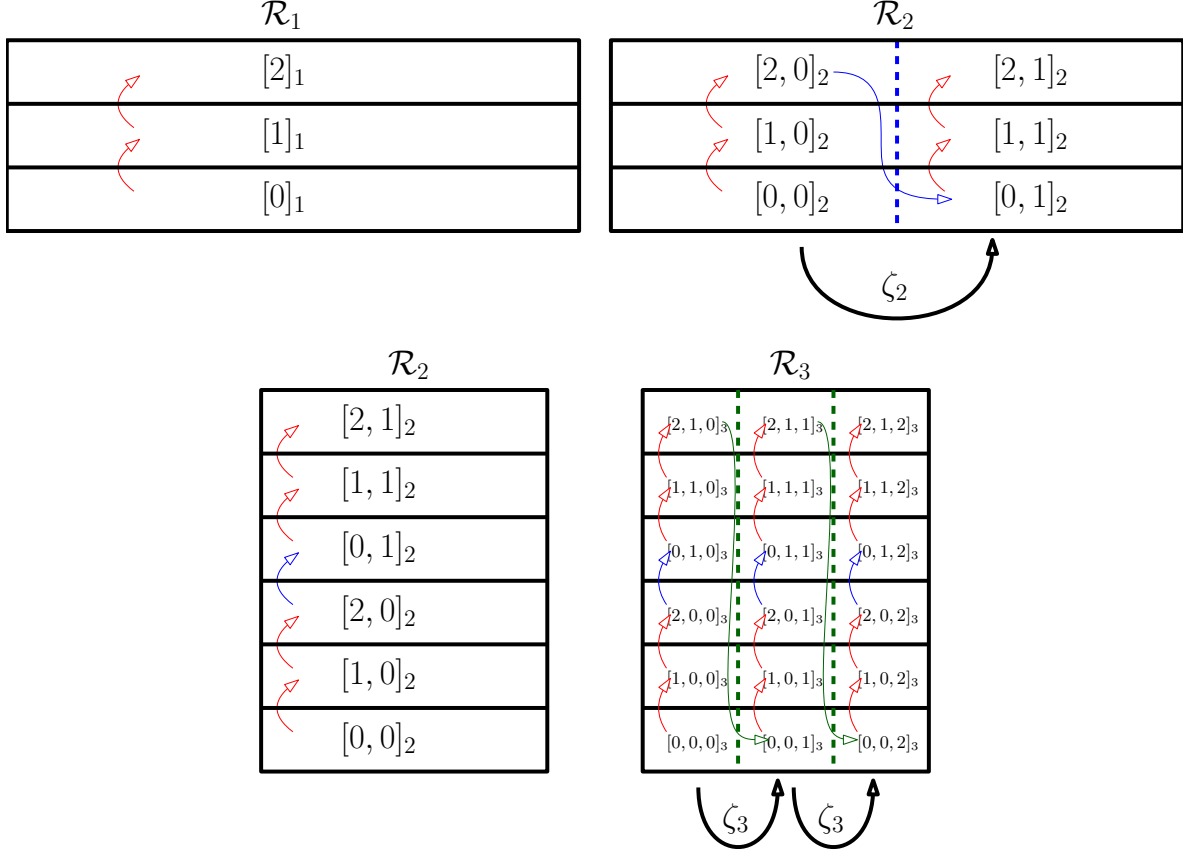


Figure 1: Example of odometer with $q_0 = 3$, $q_1 = 2$, $q_2 = 3$ (so $h_1 = 3$, $h_2 = 6$, $h_3 = 18$).

From $(q_n)_{n \geq 0}$, a new sequence $(h_n)_{n \geq 1}$ is defined by

$$\forall n \geq 1, h_n := q_0 q_1 \dots q_{n-1}.$$

The integer h_n is the height of the tower \mathcal{R}_n (see Figure 1). By convention, we set $h_0 := 1$, the height of the tower $\mathcal{R}_0 := (X)$ with a single level.

As a topological system, S is a Cantor minimal homeomorphism. As a measure-theoretic system, S is uniquely ergodic and its only invariant measure is the product $\mu := \bigotimes_{n \geq 0} \mu_n$ where μ_n is the uniform distribution on $\{0, 1, \dots, q_n - 1\}$. For the sake of completeness, we give a proof of the following well-known fact on odometers, which shows that the point spectrum is also fully understood.

Proposition 2.10. *Let S be the odometer on $\prod_{n \geq 0} \{0, 1, \dots, q_n - 1\}$. Its point spectrum is*

$$\text{Sp}(S) = \left\{ \exp \left(\frac{2i\pi k}{h_n} \right) \mid n \geq 1, 0 \leq k \leq h_n - 1 \right\}$$

and for every $\lambda = \exp \left(\frac{2i\pi k}{h_n} \right) \in \text{Sp}(S)$, the map

$$f_\lambda : x \in X \mapsto \sum_{j=0}^{h_n-1} \lambda^j \mathbf{1}_{S^j([0, \dots, 0]_n)}(x)$$

is an eigenfunction associated to λ . Moreover S has discrete spectrum.

Remark 2.11. The definition of f_λ does not depend on the choice of k and n such that $\lambda = \exp\left(\frac{2i\pi k}{h_n}\right)$. Moreover, for $n = 0$, we have $f_1 = \mathbf{1}_X$ (by convention, the 0-cylinder is X).

Proof of Proposition 2.10. Let us set $\Lambda := \left\{ \exp\left(\frac{2i\pi k}{h_n}\right) \mid n \geq 1, 0 \leq k \leq h_n - 1 \right\}$. It is straightforward to check that f_λ is an eigenfunction associated to λ , for every $\lambda \in \Lambda$. Let us show that the span of $\{f_\lambda \mid \lambda \in \Lambda\}$ is dense in $L^2(X, \mu)$. It will imply that S has discrete spectrum and that $\Lambda = \text{Sp}(S)$.

Let $n \geq 1$ and $\lambda = \exp\left(\frac{2i\pi}{h_n}\right)$. Given $a_0, \dots, a_{h_n-1} \in \mathbb{C}$, we have

$$\sum_{\ell=0}^{h_n-1} a_\ell f_{\lambda^\ell} = \sum_{j=0}^{h_n-1} P(\lambda^j) \mathbf{1}_{S^j([0, \dots, 0]_n)}$$

with the polynomial $P = a_0 + a_1 Y + \dots + a_{h_n-1} Y^{h_n-1}$. For every $j \in \{0, \dots, h_n - 1\}$, there exists a polynomial P_j of degree less than h_n , satisfying $P_j(\lambda^j) = 1$ and $P_j(\lambda^k) = 0$ for all $k \in \{0, \dots, h_n - 1\} \setminus \{j\}$. This implies that the characteristic functions of cylinders are linear combinations of the eigenfunctions f_λ for $\lambda \in \Lambda$, hence the result. \square

Let us now explain the classification of odometers up to conjugacy (and even flip-conjugacy). Let Π denote the set of prime numbers.

Definition 2.12. A **supernatural number** is a formal product of the form $\prod_{p \in \Pi} p^{k_p}$, with $k_p \in \mathbb{N} \cup \{+\infty\}$.

Given a prime number $p \in \Pi$, denote by $\nu_p(k)$ the p -adic valuation of a positive integer k . To every odometer defined with integers q_0, q_1, \dots , we associate a supernatural number $\prod_{p \in \Pi} p^{k_p}$ defined by

$$k_p := \sum_{n \geq 0} \nu_p(q_n).$$

As a consequence of Proposition 2.10 and the Halmos-von Neumann theorem, the supernatural number $\prod_{p \in \Pi} p^{k_p}$ forms a total invariant of measure-theoretic conjugacy in the class of odometers. If $k_p = \infty$ for every prime number p , then the odometer is said to be **universal**. Given a prime number p , the **p -odometer** is the odometer such that $k_p = \infty$ and $k_q = 0$ for every $q \in \Pi \setminus \{p\}$. In the case $p = 2$, it is also called the **dyadic** odometer.

Proposition 2.10 also implies that every odometer is coalescent.

Definition 2.13. A transformation $S \in \text{Aut}(X, \mu)$ is coalescent if every system $T \in \text{Aut}(X, \mu)$ which is isomorphic to S satisfies the following: every factor map from T to S is an isomorphism.

The fact that odometers are coalescent is proven in [HP68] and [New71]. In these articles, one proves that more general systems are coalescent and the phenomenon can be generalized in the context of group actions (see [IT16]). Here we give a short proof for ergodic systems with discrete spectrum.

Theorem 2.14. *Every ergodic system with discrete spectrum is coalescent.*

Proof of Theorem 2.14. Let $S \in \text{Aut}(X, \mu)$ be an ergodic system with discrete spectrum, $T \in \text{Aut}(X, \mu)$ isomorphic to S , and $\Psi: X \rightarrow X$ a factor map from T to S . Given $\lambda \in \mathbb{T}$, let us denote by $E_S(\lambda)$ (resp. $E_T(\lambda)$) the eigenspace of S (resp. T) associated to λ . First, ergodicity

implies that non-zero eigenspaces have dimension 1 (see Proper Value Theorem in [Hal56, page 34]). Secondly, since Ψ is a factor map, every eigenfunction f of S gives rise to the eigenfunction $f \circ \Psi$ of T , and more precisely $f \circ \Psi$ lies in $E_T(\lambda)$ if f lies in $E_S(\lambda)$. Hence, since S and T are isomorphic, these two remarks imply that $E_T(\lambda) = \{f \circ \Psi \mid f \in E_S(\lambda)\}$ for every λ in the point spectrum of S (or equivalently the point spectrum of T). This implies

$$L^2(X, \mu) = \{f \circ \Psi \mid f \in L^2(X, \mu)\}$$

since they have discrete spectrum. Hence Ψ is an isomorphism. \square

For the proof of Theorem D, the systems that we will consider will be an odometer S and an associated odomutant T (the odomutants are introduced in Section 3.1). Since the odomutants are extensions of their associated odometer and since we explicitly know a factor map ψ between them (see Proposition 3.4), Theorem 2.14 will ensure that we will not build an orbit equivalence between flip-conjugate systems if ψ is not invertible.

Finally, odometers have the following properties.

Proposition 2.15. *Odometers have zero measure-theoretic and topological entropies.*

Proposition 2.16. *Odometers are loosely Bernoulli.*⁷

Remark 2.17. In the case of odometers, we can notice in the following proofs that zero entropy and loose Bernoullicity follow from a poor dynamics of these systems. Indeed, given concrete partitions (for instance the partitions $\mathcal{P}(k)$ given by the cylinders of length k , which increase to the σ -algebra), the dynamics of an odometer does not generate a lot of words and the different futures are close (in the sense of the definition of loose Bernoullicity). The idea behind the definition of odomutants will be to get systems with a less "laconic" dynamics.

Proof of Proposition 2.15. Let S be an odometer. The equality $h_\mu(S) = h_{\text{top}}(S)$ follows from unique ergodicity and the variational principle (Theorem 2.4). Let $\mathcal{P}(k)$ be the partition given by the cylinders of length k . The odometer S acts as a cyclic permutation on the elements of $\mathcal{P}(k)$, so the sequence $((\mathcal{P}(k))_0^{n-1})_{n \geq 1}$ of partitions is stationary and we have $h_\mu(S, \mathcal{P}(k)) = 0$. The sequence $(\mathcal{P}(k))_{k \geq 0}$ increases to the σ -algebra of X , so we have $h_\mu(S, \mathcal{P}(k)) \xrightarrow{k \rightarrow +\infty} h_\mu(S)$ by Theorem 2.1, and we get $h_\mu(S) = 0$. \square

Proof of Proposition 2.16. Let S be an odometer, associated to the integers q_0, q_1, \dots , let $\mathcal{P}(k)$ be the partition given by the cylinders of length k . We prove that $(S, \mathcal{P}(k))$ is loosely Bernoulli for every $k \geq 1$, and we deduce from this that (S, \mathcal{P}) is loosely Bernoulli for any finite partition \mathcal{P} . We use the characterisation provided by Theorem 2.8.

Let us prove that $(S, \mathcal{P}(k))$ is loosely Bernoulli. Let $\varepsilon > 0$, $N \geq 2h_k/\varepsilon$ and $\mathcal{H} = \mathcal{P}_1^N$. Let us denote by W the word $(S^i([0, \dots, 0]_k))_{0 \leq i \leq h_k-1} \in (\mathcal{P}(k))^{\{0, \dots, h_k-1\}}$ of length h_k , this is the enumeration of the k -cylinders, with the order given by the dynamics of S . For every $x \in X$, the word $[\mathcal{P}(k)]_{1,N}(x)$ consists of the tail of the word W , followed by many concatenations of W , and the beginning of W . So any two words $w = [\mathcal{P}(k)]_{1,N}(x)$ and $w' = [\mathcal{P}(k)]_{1,N}(x')$ satisfy $f_N(w, w') \leq 2h_k/N \leq \varepsilon$. This proves that $(S, \mathcal{P}(k))$ is loosely Bernoulli.

Now let \mathcal{P} be a finite measurable partition and let us show that (S, \mathcal{P}) is loosely Bernoulli. The sequence $(\mathcal{P}(k))_{k \geq 0}$ increases to the σ -algebra of X , so for a given $\varepsilon > 0$, there exists $k \geq 0$ such that \mathcal{P} and $\mathcal{P}(k)$ are close, meaning that there exists a $\mathcal{P}(k)$ -measurable partition \mathcal{Q} , with

⁷More generally, rank-one systems are loosely Bernoulli, this is proven by Ornstein, Rudolph and Weiss [ORW82] (see Lemma 8.1) and we present their proof in the special case of odometers.

$|\mathcal{Q}| = |\mathcal{P}| = n$, and a good enumeration of the atoms of \mathcal{Q} and \mathcal{P} such that $\sum_{i=j}^n \mu(P_j \Delta Q_j) < \varepsilon$. Since $\mathcal{P}(k)$ refines \mathcal{Q} , words with respect to $\mathcal{P}(k)$ completely determine words with respect to \mathcal{Q} , so (S, \mathcal{Q}) is immediately loosely Bernoulli. Then, if N is sufficiently large, there exists $\mathcal{H} \subset \mathcal{Q}_1^N$ covering at least $1 - \varepsilon$ of the space and such that any two words $w, w' \in [\mathcal{Q}]_{1,N}(\mathcal{H})$ satisfy $f_N(w, w') \leq \varepsilon$ (the f -metric with respect to \mathcal{Q}). By the ergodic theorem, for every sufficiently large integer $N > 0$, there exists a subset X_0 of X such that $\mu(X_0) \geq 1 - \varepsilon$ and every $x \in X_0$ satisfies

$$\frac{1}{N} \left| \left\{ i \in \{1, 2, \dots, N\} \mid S^i x \in \bigcup_{j=1}^n (P_j \cap Q_j) \right\} \right| \geq 1 - 2\varepsilon.$$

This implies that for every $x \in X_0$, the word $[\mathcal{Q}]_{1,N}(x)$ determines at least a fraction $1 - 2\varepsilon$ of the word $[\mathcal{P}]_{1,N}(x)$. Therefore, given $x, x' \in X_0 \cap (\bigcup_{C \in \mathcal{H}} C)$, the words $w = [\mathcal{P}]_{1,N}(x)$ and $w' = [\mathcal{P}]_{1,N}(x')$ satisfy $f_N(w, w') \leq 5\varepsilon$ (the f -metric with respect to \mathcal{P}). It remains to define $\mathcal{H}' \subset \mathcal{P}_1^N$ as the set of atoms with non trivial intersection with $X_0 \cap \bigcup_{C \in \mathcal{H}} C$. It covers at least $1 - 3\varepsilon$ of the space and, with respect to \mathcal{P} , every two N -words w and w' produced in \mathcal{H}' satisfy $f_N(w, w') \leq 5\varepsilon$, so we are done. \square

2.6 Orbit equivalence

The conjugacy problem in full generality is very complicated (see [FRW11]). We now give the formal definition of orbit equivalence, which is a weakening of the conjugacy problem.

Definition 2.18. Two aperiodic transformations $S \in \text{Aut}(X, \mu)$ and $T \in \text{Aut}(Y, \nu)$ are **orbit equivalent** if there exists a bimeasurable bijection $\Psi: X \rightarrow Y$ satisfying $\Psi_* \mu = \nu$, such that $\text{Orb}_S(x) = \text{Orb}_{\Psi^{-1}T\Psi}(x)$ for almost every $x \in X$. The map Ψ is called an **orbit equivalence** between S and T .

We can then define the **cocycles** associated to this orbit equivalence. These are measurable functions $c_S: X \rightarrow \mathbb{Z}$ and $c_T: Y \rightarrow \mathbb{Z}$ defined almost everywhere by

$$Sx = \Psi^{-1}T^{c_S(x)}\Psi(x) \text{ and } Ty = \Psi S^{c_T(y)}\Psi^{-1}(y)$$

($c_S(x)$ and $c_T(y)$ are uniquely defined by aperiodicity).

Remark 2.19. Conversely, the existence of a cocycle, let us say c_T , implies the inclusion of the $(\Psi^{-1}T\Psi)$ -orbits in the S -orbits. So the existence of both cocycles c_S and c_T implies equality of orbits. This well-known characterization of orbit equivalence will be used in the proof of Theorem 3.16.

Given a map $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, a measurable function $f: X \rightarrow \mathbb{Z}$ is said to be **φ -integrable** if

$$\int_X \varphi(|f(x)|) d\mu < +\infty.$$

For example, integrability is exactly φ -integrability when φ is non-zero and linear, and a weaker quantification on cocycles is the notion of φ -integrability for a sublinear map φ , meaning that $\lim_{t \rightarrow +\infty} \varphi(t)/t = 0$. Two transformations in $\text{Aut}(X, \mu)$ are said to be **φ -integrably orbit equivalent** if there exists an orbit equivalence between them whose associated cocycles are φ -integrable. The notion of **L^p orbit equivalence** refers to the map $\varphi: x \rightarrow x^p$, and a **$L^{<p}$ orbit equivalence** is by definition an orbit equivalence which is L^q for all $q < p$.

Another form of quantitative orbit equivalence is Shannon orbit equivalence. We say that a measurable function $f: X \rightarrow \mathbb{Z}$ is **Shannon** if the associated partition $\{f^{-1}(n) \mid n \in \mathbb{Z}\}$ of X has finite entropy, namely

$$-\sum_{n \in \mathbb{Z}} \mu(f^{-1}(n)) \log \mu(f^{-1}(n)) < +\infty.$$

Two transformations in $\text{Aut}(X, \mu)$ are **Shannon orbit equivalent** if there exists an orbit equivalence between them whose associated cocycles are Shannon.

Note that orbit equivalence preserves ergodicity. The next statement specifically connects orbit equivalence and unique ergodicity. Theorem C and this proposition together with the variational principle directly imply Theorem B.

Proposition 2.20. *Assume that two aperiodic measurable bijections S and T on a Borel space X are orbit equivalent in the following stronger way: S and T are defined on the whole X and the equality $\text{Orb}_S(x) = \text{Orb}_T(x)$ holds for every $x \in X$.⁸ Then S is uniquely ergodic if and only if T is uniquely ergodic. In this case, S and T have the same invariant probability measure.*

Proof of Proposition 2.20. Assume that S is uniquely ergodic and denote by μ its only invariant probability measure. The cocycle $c_S: X \rightarrow \mathbb{Z}$ is defined on the whole X and is measurable. Let ν be a T -invariant probability measure. For every measurable set A , we have

$$\begin{aligned} \nu(S(A)) &= \sum_{k \in \mathbb{Z}} \nu(S(A \cap \{c_S = k\})) \\ &= \sum_{k \in \mathbb{Z}} \nu(T^k(A \cap \{c_S = k\})) \\ &= \sum_{k \in \mathbb{Z}} \nu(A \cap \{c_S = k\}) \\ &= \nu(A), \end{aligned}$$

so ν is S -invariant and is equal to μ . Therefore T is uniquely ergodic and μ is its only invariant probability measure. \square

For instance, strong orbit equivalence is a form of orbit equivalence, introduced in a topological framework by Giordano, Putnam and Skau [GPS95], to which Proposition 2.20 applies. The definition is the following.

Definition 2.21. Two Cantor minimal homeomorphisms (X, S) and (Y, T) are strongly orbit equivalent if there exists a homeomorphism $\Psi: X \rightarrow Y$ such that S and $\Psi^{-1}T\Psi$ have the same orbits on X and the associated cocycles each have at most one point of discontinuity.

Boyle proved in his thesis [Boy83] that strong orbit equivalence with continuous cocycles boils down to topological flip-conjugacy, namely S is topologically conjugate to T or to T^{-1} . As mentioned in the introduction, the classification of Cantor minimal homeomorphisms up to strong orbit equivalence is fully understood, with complete invariants such as the dimension group (see [GPS95], and Appendix B for a brief overview).

⁸This is stronger than asking this property up to a null set.

3 Odomutants

3.1 Definitions

Let $X := \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\}$ with integers $q_n \geq 2$, and let us recall the notation $h_n := q_0 \dots q_{n-1}$. The space X is endowed with the infinite product topology and we denote by μ the product of the uniform distributions on each $\{0, 1, \dots, q_n - 1\}$. We consider the odometer $S: X \rightarrow X$ on this space. Recall that it is defined by

$$Sx = \begin{cases} (\underbrace{0, \dots, 0}_{i \text{ times}}, x_i + 1, x_{i+1}, \dots) & \text{if } i := \min \{j \geq 0 \mid x_j \neq q_j - 1\} \text{ is finite} \\ (0, 0, 0, \dots) & \text{if } x = (q_0 - 1, q_1 - 1, q_2 - 1, \dots) \end{cases},$$

and it is a μ -preserving homeomorphism.

In this section, we introduce new systems that we call odomutants, defined from S with successive distortions of its orbits, encoded by the following maps ψ and ψ_n (for $n \geq 0$).

For every $n \geq 0$, we fix a finite sequence $(\sigma_i^{(n)})_{0 \leq i < q_{n+1}}$ of permutations of the set $\{0, 1, \dots, q_n - 1\}$, and we introduce

$$\psi_n: \begin{cases} X & \rightarrow X \\ x = (x_0, x_1, \dots) & \mapsto (\sigma_{x_1}^{(0)}(x_0), \sigma_{x_2}^{(1)}(x_1), \sigma_{x_3}^{(2)}(x_2), \dots, \sigma_{x_{n+1}}^{(n)}(x_n), x_{n+1}, x_{n+2}, \dots) \end{cases}.$$

It is not difficult to see that ψ_n is a homeomorphism and preserves the measure μ , its inverse is given by

$$\psi_n^{-1}: \begin{cases} X & \rightarrow X \\ x = (x_0, x_1, \dots) & \mapsto (z_0(x), z_1(x), \dots, z_n(x), x_{n+1}, x_{n+2}, \dots) \end{cases}$$

with $z_i(x)$ defined by backwards induction as follows:

$$\begin{aligned} z_n(x) &:= (\sigma_{x_{n+1}}^{(n)})^{-1}(x_n), \\ z_i(x) &:= (\sigma_{z_{i+1}(x)}^{(i)})^{-1}(x_i) \text{ for every } i \in \{0, 1, \dots, n-1\}. \end{aligned} \tag{1}$$

Let us also introduce

$$\psi: \begin{cases} X & \rightarrow X \\ x = (x_0, x_1, \dots) & \mapsto (\sigma_{x_{n+1}}^{(n)}(x_n))_{n \geq 0}. \end{cases}$$

The map ψ is continuous but is not invertible in full generality. It is not difficult to see that $\psi_n(x) \xrightarrow{n \rightarrow +\infty} \psi(x)$ for every $x \in X$. The map ψ also have the following properties.

Proposition 3.1. *$\psi: X \rightarrow X$ preserves the probability measure μ and is onto.*

Proof of Proposition 3.1. To prove that μ is ψ -invariant, it suffices to prove the equality $\mu(\psi^{-1}(A)) = \mu(A)$ when A is a cylinder. If A is an $(n+1)$ -cylinder, then $\psi^{-1}(A) = \psi_n^{-1}(A)$, so the ψ -invariance follows from the ψ_n -invariance for all $n \geq 0$.

Given $y \in X$, let us find $x \in X$ such that $\psi(x) = y$. By definition, for every $n \geq 0$, $\psi(\psi_n^{-1}(y))$ is in the cylinder $[y_0, \dots, y_n]_{n+1}$, so $\psi(\psi_n^{-1}(y)) \xrightarrow{n \rightarrow +\infty} y$. By compactness, there exists a convergent subsequence of $(\psi_n^{-1}(y))_{n \geq 0}$, of limit $x \in X$, and we have $\psi(x) = y$ since ψ is continuous. \square

The following computations motivate the definition of odomutants. Let us respectively set the **minimal** and **maximal** points of X :

$$x^- := (0, 0, 0, \dots) \text{ and } x^+ := (q_0 - 1, q_1 - 1, q_2 - 1, \dots).$$

We define the following sets

$$\begin{aligned} X_n^- &:= \{x \in X \mid (x_0, \dots, x_n) \neq (x_0^-, \dots, x_n^-)\}, \\ X_n^+ &:= \{x \in X \mid (x_0, \dots, x_n) \neq (x_0^+, \dots, x_n^+)\}, \\ X_\infty^- &:= X \setminus \{x^-\} \text{ and } X_\infty^+ := X \setminus \{x^+\}. \end{aligned}$$

It is not difficult to see that X_∞^+ is the increasing union of the sets X_n^+ , so for every $x \in X_\infty^+$, we denote by $N^+(x)$ the least integer $n \geq 0$ satisfying $x \in X_n^+$. This also holds for X_∞^- and X_n^- , and $N^-(x)$ is defined similarly.

Let $x \in \psi^{-1}(X_\infty^+)$ and $N := N^+(\psi(x))$. By definition of N , for every $n \geq N$, $S\psi_n(x)$ is equal to

$$(\underbrace{0, \dots, 0}_{N \text{ times}}, \sigma_{x_{N+1}}^N(x_N) + 1, \sigma_{x_{N+2}}^{(N+1)}(x_{N+1}), \dots, \sigma_{x_{n+1}}^{(n)}(x_n), x_{n+1}, x_{n+2}, \dots).$$

Using (1), we get

$$\psi_n^{-1}S\psi_n(x) = (y_0^{(n)}(x), \dots, y_n^{(n)}(x), x_{n+1}, x_{n+2}, \dots)$$

with $y_i^{(n)}(x)$ defined by backwards induction as follows:

$$\begin{aligned} y_n^{(n)}(x) &:= \left(\sigma_{x_{n+1}}^{(n)}\right)^{-1}(\sigma_{x_{n+1}}^{(n)}(x_n)) = x_n, \\ \forall n > i > N, y_i^{(n)}(x) &:= \left(\sigma_{y_{i+1}^{(n)}(x)}^{(i)}\right)^{-1}(\sigma_{x_{i+1}}^{(i)}(x_i)), \\ y_N^{(n)}(x) &:= \left(\sigma_{y_{N+1}^{(n)}(x)}^{(N)}\right)^{-1}(\sigma_{x_{N+1}}^{(N)}(x_N) + 1), \\ \forall N > i \geq 0, y_i^{(n)}(x) &:= \left(\sigma_{y_{i+1}^{(n)}(x)}^{(i)}\right)^{-1}(0). \end{aligned}$$

By induction, it is easy to get $(y_{N+1}^{(n)}(x), \dots, y_n^{(n)}(x)) = (x_{N+1}, \dots, x_n)$ and this implies the following simplification: $\psi_n^{-1}S\psi_n(x)$ is equal to $(y_0^{(n)}(x), \dots, y_N^{(n)}(x), x_{N+1}, x_{N+2}, \dots)$ with $y_i^{(n)}(x)$ inductively defined by

$$\begin{aligned} y_N^{(n)}(x) &:= \left(\sigma_{x_{N+1}}^{(N)}\right)^{-1}(\sigma_{x_{N+1}}^{(N)}(x_N) + 1), \\ \forall 0 \leq i \leq N-1, y_i^{(n)}(x) &:= \left(\sigma_{y_{i+1}^{(n)}(x)}^{(i)}\right)^{-1}(0). \end{aligned}$$

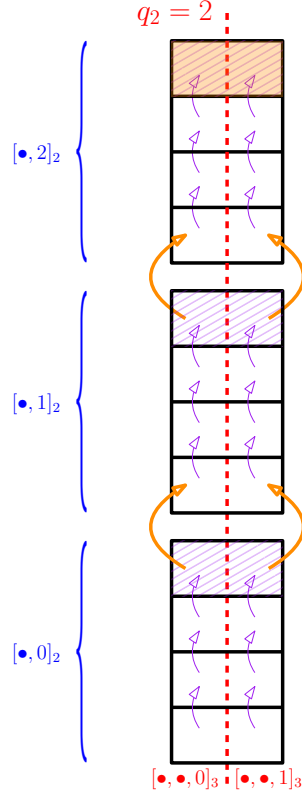
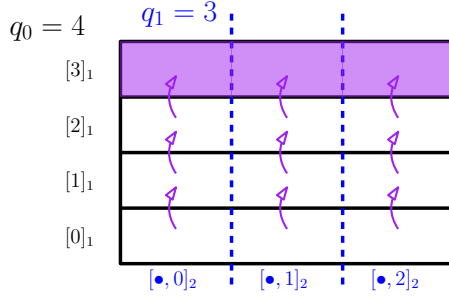
Finally, $(y_0^{(n)}(x), \dots, y_N^{(n)}(x))$ does not depend on the integer $n \geq N^+(\psi(x))$.

Definition 3.2. For every $x \in \psi^{-1}(X_\infty^+)$, let us define

$$Tx := \psi_n^{-1}S\psi_n(x)$$

for any $n \geq N^+(\psi(x))$. The map T is called the **odomotant** associated to the odometer S and the sequences of permutations $\left(\sigma_i^{(n)}\right)_{0 \leq i < q_{n+1}}$ for $n \geq 0$.

The odometer S on $\prod_{n \geq 0} \{0, \dots, q_n - 1\}$



An odomutant T associated to S

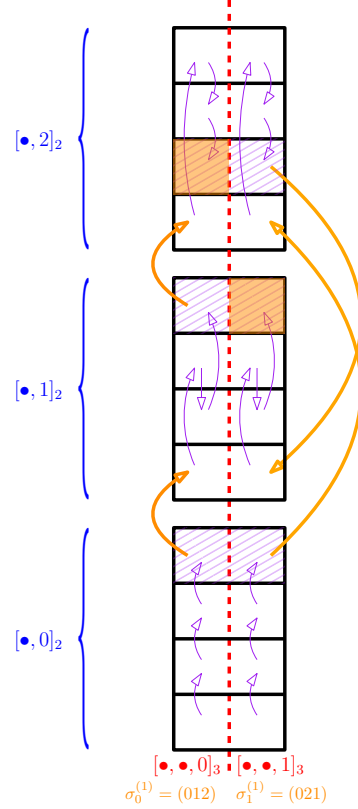
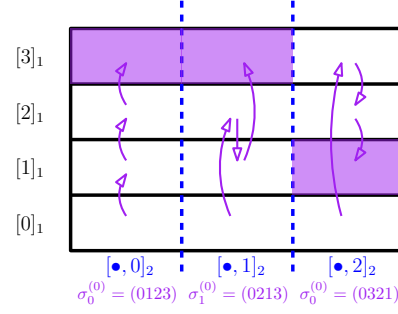


Figure 2: Example of the first two steps in the construction of an odometer (on the left) and an associated odomutant (on the right). For a permutation σ of the set $\{0, \dots, k-1\}$, the notation $\sigma = (i_0 \dots i_{k-1})$ means that σ is defined by $\sigma(j) = i_j$ for every $j \in \{0, \dots, k-1\}$. The area coloured in purple (resp. orange) is the subset on which S and T are not yet defined at the end of the first step (resp. second step), it is equal to $\{N^+ = 1\}$ (resp. $\{N^+ = 2\}$) for the odometer, $\{N^+ \circ \psi = 1\}$ (resp. $\{N^+ \circ \psi = 2\}$) for the odomutant.

As illustrated in Figure 2, an odomutant T is a probability measure-preserving bijection that we build step by step. At step n , T is well-defined on $\{N^+(x) = n\}$. This is a cutting-and-stacking method very similar to the odometer, but at every step the way we connect the subcolumns of the tower depend on the next coordinates.

3.2 Odomutants with multiplicities

At first view, when looking at Figure 2, we can think that an odomutant is encoded by a cutting-and-stacking construction where the new towers at each step are built by stacking only *one* copy of the dynamics of each subcolumn. Actually, with some redundancies in the permutations of a same step, it is possible to encode a cutting-and-stacking construction where, at every step and for every subcolumn, many copies of its dynamics could appear in each new tower (as illustrated in Figure 3). In this case, the partitions in cylinder of the same length are not the information we want to keep in mind, since they also remember that we divide the subcolumns to get many copies of its dynamics. This motivates the following definition that we explain with more details after.

Definition 3.3. Let $(q_n)_{n \geq 0}$ be a sequence of integers greater than or equal to 2. Let $\mathbf{c} = (c_{n,0}, \dots, c_{n,\tilde{q}_n-1})_{n \geq 1}$ be a sequence where \tilde{q}_n and $c_{n,i}$ are positive integers satisfying $q_n = c_{n,1} + \dots + c_{n,\tilde{q}_n}$, and $(\tau_j^{(n)})_{j \in \{0, \dots, \tilde{q}_{n+1}-1\}}$ be a sequence of permutations of the set $\{0, \dots, q_n-1\}$ for every $n \geq 0$. For every $n \geq 1$ and every $j \in \{0, \dots, \tilde{q}_n-1\}$, we set

$$I_j^{(n)} := \left(\sum_{i=0}^{j-1} c_{n,i} \right) + \{0, 1, \dots, c_{n,j}-1\}.$$

Then we say that T is the odomutant built with **\mathbf{c} -multiple permutations** $\tau_j^{(n)}$, if T is the odomutant associated to the odometer on the space $\prod_{n \geq 0} \{0, \dots, q_n-1\}$ and families of permutations $(\sigma_i^{(n)})_{0 \leq i < q_{n+1}}$, where for every $n \geq 0$ and every $j \in \{0, \dots, \tilde{q}_{n+1}\}$, we have $\sigma_i^{(n)} := \tau_j^{(n)}$ for all integers $i \in I_j^{(n+1)}$.

In this case, we associate partitions $\tilde{\mathcal{P}}(\ell)$ for every $\ell \geq 1$, defined by

$$\tilde{\mathcal{P}}(\ell) := \left\{ [i_0, \dots, i_{\ell-2}, I_j^{(\ell-1)}]_\ell \mid 0 \leq i_0 < q_0, \dots, 0 \leq i_{\ell-2} < q_{\ell-2}, 0 \leq j \leq \tilde{q}_{\ell-1}-1 \right\}.$$

We say that the odomutant is built with **uniformly \mathbf{c} -multiple permutations** if we have $c_{n,0} = \dots = c_{n,\tilde{q}_n-1} =: c_n$ for every $n \geq 1$, and we simply write $\mathbf{c} := (c_n, \tilde{q}_n)_{n \geq 0}$.

At the beginning of step n , for every $i \in \{0, \dots, \tilde{q}_n-1\}$ there are c_i subcolumns which have been defined with the same permutation¹⁰ $\tau_i^{(n-1)}$ at step $n-1$, they actually play the role of c_i copies of the dynamics of a subcolumn that we would like to stack c_i times in each tower. When considering the partition $\tilde{\mathcal{P}}(n+1)$, we cannot distinguish between these "copies", as if it was the partition made up of the subcolumns that we would like to stack more than once in each tower.

The odomutants built with uniformly multiple permutations, equipped with the associated partitions $(\tilde{\mathcal{P}}(\ell))_{\ell \geq 1}$, better describe Boyle and Handelman's constructions [BH94] than odomutants equipped with $\mathcal{P}(\ell)_{\ell \geq 1}$. We refer the reader to Appendix B for more details, more precisely in Section B.4. The sequences $(c_n)_n$ and $(\tilde{q}_n)_n$ respectively correspond to the sequences $(n_k)_k$ and $(m_k)_k$ introduced in their paper. Then, to prove Theorem C in the case $\alpha = +\infty$, we will partly reformulate the proof of their similar statement with our formalism. Our proof in the case $\alpha < +\infty$ will be different than theirs since we will build an odomutant with pairwise different permutations at each step.

⁹We write $s + \{0, 1, \dots, k\} := \{s, s+1, \dots, s+k\}$. The family $(I_0^{(n)}, \dots, I_{\tilde{q}_n-1}^{(n)})$ forms a partition of $\{0, 1, \dots, q_n-1\}$.

¹⁰Note that the permutations $\tau_0^{(n-1)}, \dots, \tau_{\tilde{q}_n-1}^{(n-1)}$ are not necessarily pairwise different.

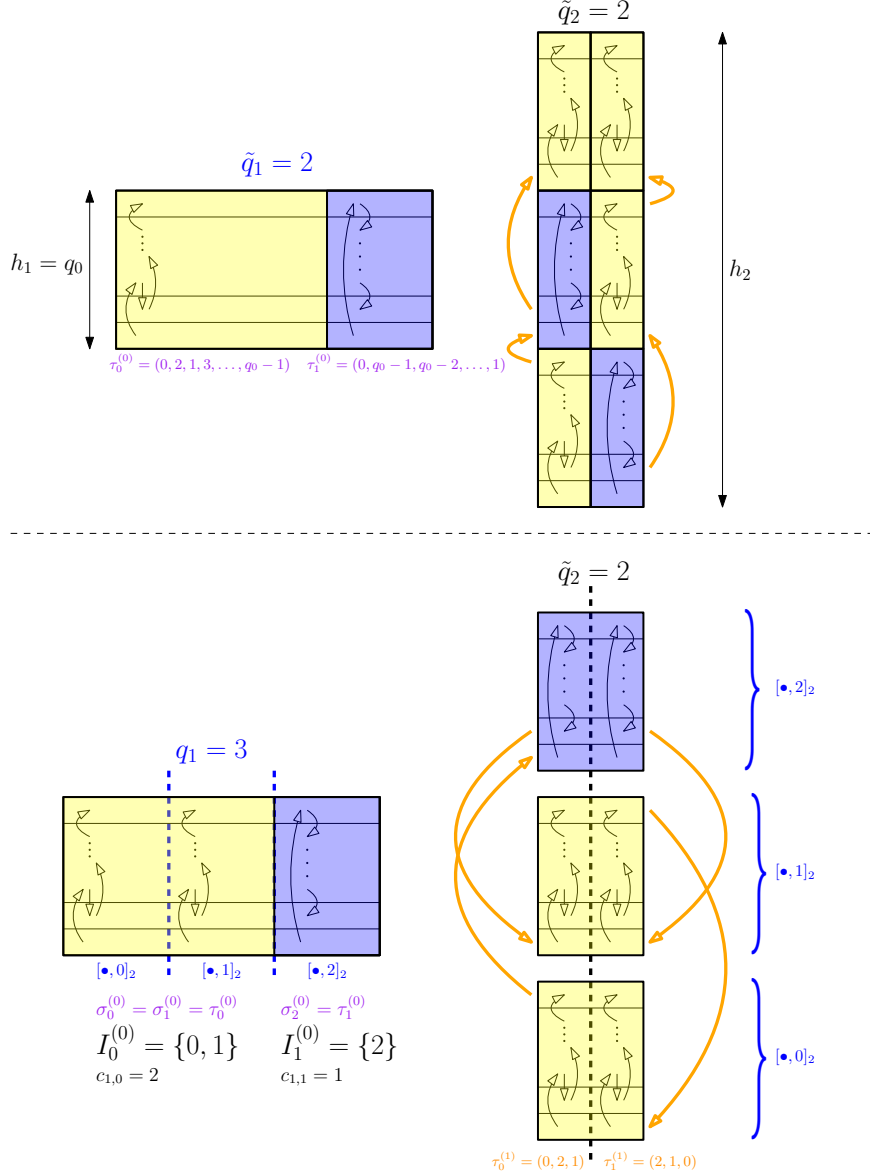


Figure 3: At the top, the second step of a less restrictive cutting-and-stacking construction that we want to describe with an odomutant. At the bottom, the way we encode it with such a system. Here, the dynamics of the yellow tower appears twice in each new tower, so we divide it in two subtowers. Note that the partition $\tilde{\mathcal{P}}(2)$ is exactly the partition which gives the colour (yellow or blue) and the level in the h_0 -tower to each points of the space, so that we cannot distinguish between points of the two yellow subtowers which are at the same level, contrary to the partition $\mathcal{P}(2)$. For the third step of the construction, the value of q_2 will depend on the number ($c_{2,0}$ and $c_{2,1}$) of copies for the dynamics of the two current towers in the next ones.

As mentionned in the introduction, our formalism of odomutants was inspired by Feldman's construction [Fel76] of a non-loosely Bernoulli system. As we will see in the proof of Theorem A, this system is an odomutant built with uniformly \mathbf{c} -multiple permutations where the integers c_n are powers of 2 and for a fixed n , the permutations $\tau_i^{(n)}$ are pairwise different at each step.

3.3 Odomutants as p.m.p. bijections on a standard probability space

In this section, we study odomutants with a measure-theoretic viewpoint.

3.3.1 First properties

Proposition 3.4. *T is a bijection from $\psi^{-1}(X_\infty^+)$ to $\psi^{-1}(X_\infty^-)$, its inverse is given by*

$$T^{-1}y = \psi_n^{-1}S^{-1}\psi_n(y)$$

for every $y \in \psi^{-1}(X_\infty^-)$ and any $n \geq N^-(\psi(y))$. Moreover T is an element of $\text{Aut}(X, \mu)$ and ψ is a factor map from T to S .

Proof of Proposition 3.4. The equality $\psi_n(Tx) = S\psi_n(x)$ implies $\psi(Tx) = S\psi(x)$ since ψ_n converges pointwise to ψ . Moreover, the map ψ preserves the measure μ and is onto (see Proposition 3.1). Thus, assuming that T is in $\text{Aut}(X, \mu)$, S is a factor of T via the factor map ψ .

Since X_∞^+ is the increasing union of the sets X_n^+ , and for every $n \geq 0$, T and $\psi_n^{-1}S\psi_n$ coincide on X_n^+ , the injectivity of T on $\psi^{-1}(X_\infty^+)$ follows from the injectivity of S and the maps ψ_n and ψ_n^{-1} .

For $x \in \psi^{-1}(X_\infty^+)$, we have $\psi(Tx) = S\psi(x)$ and $\psi(x) \neq x^+$, so $\psi(Tx)$ is not equal to x^- . Conversely, for $y \in \psi^{-1}(X_\infty^-)$, the element $x := \psi_n^{-1}S^{-1}\psi_n(y)$ does not depend on the choice of an integer $n \geq N^-(\psi(y))$ (these are the same computations as before Definition 3.2) and satisfies $Tx = y$.

By ψ -invariance, the sets $\psi^{-1}(X_\infty^+)$ and $\psi^{-1}(X_\infty^-)$ have full measure, so $T: X \rightarrow X$ is a bijection up to measure zero. It follows again from the properties of S and the maps ψ_n that T is bimeasurable and preserves the measure μ . \square

The next result provides a criterion for ψ to be an isomorphism between T and S . We will not apply it in this paper but it enables us to understand that, in case permutations have common fixed points¹¹ (see Section 3.5), we will need the sequence $(q_n)_{n \geq 0}$ to increase quickly enough, otherwise we get an odomutant T conjugate to S .

Lemma 3.5. *For every $n \geq 0$, we set*

$$F_n := \{x_n \in \{0, \dots, q_n - 1\} \mid \forall x_{n+1} \in \{0, \dots, q_{n+1} - 1\}, \sigma_{x_{n+1}}^{(n)}(x_n) = x_n\}.$$

If the series $\sum \frac{|F_n|}{q_n}$ diverges, then ψ is an isomorphism between S and T .

Proof of Lemma 3.5. By the Borel-Cantelli lemma, the set

$$X_0 := \{(x_n)_{n \geq 0} \in X \mid x_n \in F_n \text{ for infinitely many integers } n\}$$

has full measure. It is also S -, T - and ψ -invariant and it is easy to check that $\psi: X_0 \rightarrow X_0$ is a bijection, using the fact that the equality $\sigma_{x_{n+1}}^{(n)}(x_n) = y_n$ implies $x_n = y_n$ when y_n is in F_n . \square

Remark 3.6. It is not hard to see, independently of Lemma 3.5, that in order to prove Theorems A and B, one needs the sequence $(q_n)_{n \geq 0}$ to be unbounded. Otherwise, let K denote an upper bound of the sequence, then the underlying odomutant admits a cutting-and-stacking construction with at most K towers at each step. A system satisfying such property is said to have *rank K* (and more generally *finite rank*) and it is well-known that it is loosely Bernoulli and has zero entropy (see [Fer97]).

¹¹For Theorem A (resp. Theorem C), we will require $\sigma_i^{(n)}(0) = 0$ (resp. $\sigma_i^{(n)}(0) = 0$ and $\sigma_i^{(n)}(q_n - 1) = q_n - 1$).

Question 3.7. Is it possible to find a necessary and sufficient condition on the permutations $\sigma_i^{(n)}$ (for $n \geq 0$ and $0 \leq i < q_{n+1}$) for the factor map ψ to be an isomorphism? Since every odometer is coalescent (see Theorem 2.14), this would enable us to know whether or not an odomutant is conjugate to its associated odometer.

The following two results will be useful for some computations in the proofs of Lemma 3.11 and Proposition 3.15. They deal with the well-definedness of powers (positive or negative) of an odomutant at some point of X .

Proposition 3.8. *For $k \in \mathbb{N}$, the following assertion hold.¹²*

- If $\psi(x)$ is in $\bigcap_{i=0}^{k-1} S^{-i}(X_\infty^+)$, then Tx, T^2x, \dots, T^kx are well-defined and for every $i \in \{0, \dots, k\}$, we have

$$T^i x = \psi_n^{-1} S^i \psi_n(x)$$

for any $n \geq \max_{0 \leq j \leq i-1} N^+(\psi(T^j x))$.

- If $\psi(x)$ is in $\bigcap_{i=-(k-1)}^0 S^{-i}(X_\infty^-)$, then $T^{-1}x, T^{-2}x, \dots, T^{-k}x$ are well-defined and for every $i \in \{-(k-1), \dots, 0\}$, we have

$$T^{-i} x = \psi_n^{-1} S^{-i} \psi_n(x)$$

for any $n \geq \max_{-(i-1) \leq j \leq 0} N^-(\psi(T^j x))$.

Proof of Proposition 3.8. For example, let us prove the first point by induction over $k \geq 1$. The proof of the second point is similar.

The result is clear for $k = 0$. Let $k \geq 1$. Let us assume that the result holds for $k - 1$ and that

$$\psi(x) \in \bigcap_{i=0}^{k-1} S_n^{-i}(X_\infty^+).$$

This implies that $T^{k-1}x$ is well-defined and is equal to $\psi_n^{-1} S^{k-1} \psi_n(x)$ for any n greater than or equal to $\max_{0 \leq j \leq k-2} N^+(\psi(T^j x))$. Moreover $\psi(T^{k-1}x)$ is not equal to x^+ . Indeed, the first $n + 1$ coordinates of $\psi(T^{k-1}x)$ and $\psi_n(T^{k-1}x)$ are the same and we have

$$\psi_n(T^{k-1}x) = S^{k-1} \psi_n(x)$$

for any $n \geq \max_{0 \leq j \leq k-2} N^+(\psi(T^j x))$, so this follows from the fact that $S^{k-1} \psi(x)$ is not equal to x^+ . This implies that $T^k x$ is well-defined and equal to $\psi_n^{-1} S \psi_n(T^{k-1}x)$ for any $n \geq N^+(\psi(T^{k-1}x))$. Finally, for any $n \geq \max_{0 \leq j \leq k-1} N^+(\psi(T^j x))$, we get

$$T^k x = \psi_n^{-1} S \psi_n(T^{k-1}x) = \psi_n^{-1} S \psi_n(\psi_n^{-1} S^{k-1} \psi_n(x)) = \psi_n^{-1} S^k \psi_n(x),$$

hence the result for k . □

Corollary 3.9. *Let $x, y \in X$ and $M \in \mathbb{N}^*$ such that $x_j = y_j$ for every $j \geq M$, and set*

$$K := \sum_{j=0}^{M-1} h_j \left(\sigma_{y_{j+1}}^{(j)}(y_j) - \sigma_{x_{j+1}}^{(j)}(x_j) \right).$$

Assume that x and y are different. Then the following hold:

¹²For instance, this holds for every $x \in X_\infty^+$ such that $\psi(x)$ is not in $\text{Orb}_S(x^+)$ (which is also the S -orbit of x^-), so the hypothesis holds for a set of points x of full measure.

- if $K > 0$, then Tx, T^2x, \dots, T^Kx are well-defined;
- if $K < 0$, then $T^{-1}x, T^{-2}x, \dots, T^Kx$ are well-defined.

Moreover we have $T^Kx = y$.

Remark 3.10. The proof of the equality $T^Kx = y$ is based on the well-understood case of an odometer, namely the permutations $\sigma_i^{(n)}$ are all identity maps and $T = S$. More precisely, given $w, z \in X$ satisfying $w_j = z_j$ for every j greater than or equal to some M , we know that $S^Kw = z$ with

$$K = \sum_{j=0}^{M-1} h_j (z_j - w_j).$$

It remains to apply this well-known fact to $w = \psi_n(x)$ and $z = \psi_n(y)$ for a large enough integer n .

Proof of Corollary 3.9. Let us consider the case $K > 0$ (the proof for the other case is similar). By the previous remark, it is clear that we have

$$y = \psi_n^{-1} S^K \psi_n(x)$$

for every $n \geq M$. Using Proposition 3.8, it remains to prove that $S^i \psi(x)$ is not equal to x^+ for every $i \in \{0, \dots, K-1\}$. If there exists a positive integer i such that $S^i \psi(x) = x^+$, then we have

$$\sigma_{x_{j+1}}^{(j)}(x_j) = q_j - 1$$

for every sufficiently large integers j , and

$$\begin{aligned} i &= \sum_{j=0}^{+\infty} h_j \left(q_i - 1 - \sigma_{x_{j+1}}^{(j)}(x_j) \right) \\ &= \sum_{j=0}^{M-1} h_j \left(q_i - 1 - \sigma_{x_{j+1}}^{(j)}(x_j) \right) + \sum_{j=M}^{+\infty} h_j \left(q_i - 1 - \sigma_{x_{j+1}}^{(j)}(x_j) \right) \\ &\geq \sum_{j=0}^{M-1} h_j \left(\sigma_{y_{j+1}}^{(j)}(y_j) - \sigma_{x_{j+1}}^{(j)}(x_j) \right). \end{aligned}$$

Therefore i is greater than or equal to K and we are done. \square

3.3.2 An odomutant and its associated odometer have the same point spectrum

Since every odomutant T factors onto its associated odometer S , we have the inclusion $\text{Sp}(S) \subset \text{Sp}(T)$ between the point spectrums. We actually show that this is an equality. The following lemma is inspired by Danilenko and Vieprik's methods to study the point spectrum of rank-one systems (see Proposition 3.7 in [DV23]).

Lemma 3.11. *Let T be an odomutant built from the odometer S on $X = \prod_{n \geq 0} \{0, \dots, q_n - 1\}$ and the families of permutations $(\sigma_{x_{n+1}}^{(n)})_{0 \leq x_{n+1} < q_{n+1}}$. If $\lambda \in \mathbb{T}$ is an eigenvalue of T , then for every $\varepsilon > 0$, there exists a positive integer n such that for every $m \geq n$, there exist $E_{n,m} \subset \prod_{j=n}^m \{0, \dots, q_j - 1\}$ and $x_{m+1} \in \{0, \dots, q_{m+1} - 1\}$ satisfying the following:*

- $\frac{|E_{n,m}|}{q_n q_{n+1} \dots q_m} > 1 - \varepsilon;$

- for every $(y_n, \dots, y_m), (z_n, \dots, z_m) \in E_{n,m}$, we have

$$\left| 1 - \lambda^{\sum_{j=n}^m h_j (\sigma_{y_{j+1}}^{(j)}(y_j) - \sigma_{z_{j+1}}^{(j)}(z_j))} \right| < \varepsilon,$$

with $y_{m+1} = z_{m+1} := x_{m+1}$.

Proof of Lemma 3.11. Let $\varepsilon > 0$, $\lambda \in \mathbb{T}$ an eigenvalue of T and g_λ an eigenfunction of T associated to λ . Without loss of generality, we assume that $\varepsilon \leq 1/2$. Moreover, the modulus of g_λ is almost everywhere constant (since it is T -invariant and T is ergodic), so we assume that g_λ has modulus 1. There exists $\alpha \in \mathbb{T}$ and a measurable subset $A \subset X$ of positive measure such that

$$\forall x \in A, |g_\lambda(x) - \alpha| < \varepsilon/2. \quad (2)$$

Since the partition given by the n -cylinders is increasing to the σ -algebra on X as $n \rightarrow +\infty$, we can find $n > 0$ and $(x_0, \dots, x_{n-1}) \in \prod_{j=0}^{n-1} \{0, \dots, q_j - 1\}$ such that

$$\mu(A \cap [x_0, \dots, x_{n-1}]_n) > (1 - \varepsilon^2)\mu([x_0, \dots, x_{n-1}]_n).$$

Let $m \geq n$. Then there exists $x_{m+1} \in \{0, \dots, q_{m+1} - 1\}$ such that

$$\mu(A \cap [x_0, \dots, x_{n-1}, \bullet, \dots, \bullet, x_{m+1}]_{m+2}) > (1 - \varepsilon^2)\mu([x_0, \dots, x_{n-1}, \bullet, \dots, \bullet, x_{m+1}]_{m+2}) \quad (3)$$

and we set

$$E_{n,m} := \left\{ (y_n, \dots, y_m) \in \prod_{j=n}^m \{0, \dots, q_j - 1\} \mid \frac{\mu(A \cap [x_0, \dots, x_{n-1}, y_n, \dots, y_m, x_{m+1}]_{m+2})}{(1 - \varepsilon)\mu([x_0, \dots, x_{n-1}, y_n, \dots, y_m, x_{m+1}]_{m+2})} > 1 - \varepsilon \right\}.$$

By Inequality (3), we get

$$\frac{|E_{n,m}|}{q_n \cdots q_m} > 1 - \varepsilon.$$

Let $(y_n, \dots, y_m), (z_n, \dots, z_m) \in E_{n,m}$. Let us set

$$B_y := A \cap [x_0, \dots, x_{n-1}, y_n, \dots, y_m, x_{m+1}]_{m+2},$$

$$B_z := A \cap [x_0, \dots, x_{n-1}, z_n, \dots, z_m, x_{m+1}]_{m+2}.$$

and

$$K := \sum_{j=n}^m h_j \left(\sigma_{y_{j+1}}^{(j)}(y_j) - \sigma_{z_{j+1}}^{(j)}(z_j) \right)$$

(with $y_{m+1} = z_{m+1} := x_{m+1}$). By Corollary 3.9, the set $T^{-K}(B_y)$ is included in the cylinder

$$C := [x_0, \dots, x_{n-1}, z_n, \dots, z_m, x_{m+1}]_{m+2},$$

which implies that $B := T^{-K}(B_y) \cap B_z$ has positive measure. Indeed, if B were a null set, the cylinder C would contain two subsets $T^{-K}(B_y)$ and B_z of negligible intersection and we would get $\mu(C) > 2(1 - \varepsilon)\mu(C)$ by definition of $E_{n,m}$, this is not possible since $\varepsilon \leq 1/2$.

Then we have $g_\lambda(T^K x) = \lambda^K g_\lambda(x)$ for almost every $x \in B$, and since every $x \in B$ is in A and satisfies $T^K x \in A$, we get $|1 - \lambda^K| < \varepsilon$ using (2). \square

Lemma 3.12. *Let $0 < \varepsilon < 2$ and $\theta = \theta(\varepsilon) > 0$ such that*

$$\{\nu \in \mathbb{T} \mid |1 - \nu| < \varepsilon\} = \{\exp(2i\pi\tau) \mid -\theta < \tau < \theta\}.$$

Let $\nu \in \mathbb{T} \setminus \{1\}$ satisfying $|1 - \nu| < \varepsilon$. We write it as $\nu = \exp(2i\pi\tau)$ with $-\theta < \tau < \theta$, $\tau \neq 0$. If ε is small enough so that $\theta < 1/4$, then for every interval¹³ J of \mathbb{Z} , we have

$$\sum_{j \in J} \mathbf{1}_{|1 - \nu^j| < \varepsilon} \leq \frac{3\theta}{1 - 2\theta} |J| + \frac{6\theta}{|\tau|}.$$

Proof of Lemma 3.12. Without loss of generality, we assume that τ is positive. Let J be an interval of \mathbb{Z} . If we have

$$\sum_{j \in J} \mathbf{1}_{|1 - \nu^j|} = 0,$$

then the result is clear. Now we assume that there exists $j \in J$ such that $|1 - \nu^j| < \varepsilon$. Since ν is not equal to 1, this implies that we have $|1 - \nu^k| < \varepsilon$ for infinitely many integers k . Since θ is less than $1/4$, we also have $|1 - \nu^k| \geq \varepsilon$ for infinitely many integers k . Therefore we can find sequences $(n_\ell)_{\ell \in \mathbb{Z}}$ and $(m_\ell)_{\ell \in \mathbb{Z}}$ of integers such that $n_\ell < m_\ell < n_{\ell+1} < m_{\ell+1}$ for every $\ell \in \mathbb{Z}$, so that we can write

$$\mathbb{Z} = \dots \sqcup C_{-2} \sqcup D_{-2} \sqcup C_{-1} \sqcup D_{-1} \sqcup C_0 \sqcup D_0 \sqcup C_1 \sqcup D_1 \sqcup \dots$$

with intervals $C_\ell := \{k \in \mathbb{Z} \mid n_\ell \leq k < m_\ell\}$ and $D_\ell := \{k \in \mathbb{Z} \mid m_\ell \leq k < n_{\ell+1}\}$ such that

$$\forall k \in C_\ell, |1 - \nu^k| < \varepsilon \text{ and } \forall k \in D_\ell, |1 - \nu^k| \geq \varepsilon.$$

For every $\ell \in \mathbb{Z}$, we have

$$(m_\ell - n_\ell - 1)\tau < 2\theta < (m_\ell - n_\ell + 1)\tau$$

$$\text{and } (n_{\ell+1} - m_\ell + 1)\tau \geq 1 - 2\theta,$$

this implies

$$\frac{2\theta}{\tau} - 1 < |C_\ell| < \frac{2\theta}{\tau} + 1$$

$$\text{and } |D_\ell| \geq \frac{1 - 2\theta}{\tau} - 1.$$

Now we set $\ell_0 := \max\{\ell \in \mathbb{Z} \mid n_\ell \leq \min J\}$ and $\ell_1 := \max\{\ell \in \mathbb{Z} \mid n_\ell \leq \max J\}$. We then have the inclusion $\bigsqcup_{\ell_0+1 \leq \ell \leq \ell_1-1} (C_\ell \sqcup D_\ell) \subset J$ which yields

$$|J| > (\ell_1 - \ell_0 - 1) \left(\frac{1}{\tau} - 2 \right).$$

¹³By an interval of \mathbb{Z} , we mean a set of the form $\{k \in \mathbb{Z} \mid a \leq k \leq b\}$ for some integers a and b .

Finally, we have

$$\begin{aligned}
\sum_{j \in J} \mathbb{1}_{|1-\nu^j| < \varepsilon} &\leq \sum_{\ell=\ell_0}^{\ell_1} |C_\ell| \\
&\leq (\ell_1 - \ell_0 + 1) \left(\frac{2\theta}{\tau} + 1 \right) \\
&\leq \left(\frac{\frac{2\theta}{\tau} + 1}{\frac{1}{\tau} - 2} \right) |J| + 2 \left(\frac{2\theta}{\tau} + 1 \right) \\
&\leq \left(\frac{\frac{2\theta}{\tau} + \frac{\theta}{\tau}}{\frac{1}{\tau} - 2\frac{\theta}{\tau}} \right) |J| + 2 \left(\frac{2\theta}{\tau} + \frac{\theta}{\tau} \right) \text{ since } 1 \leq \frac{\theta}{\tau} \\
&= \frac{3\theta}{1-2\theta} |J| + \frac{6\theta}{\tau}
\end{aligned}$$

and we are done. \square

Theorem 3.13. *Let T be an odomutant built from the odometer S on $X = \prod_{n \geq 0} \{0, \dots, q_n - 1\}$. Then T and S have the same point spectrum.*

Using the Halmos-von Neumann theorem [HVN42], we get the following corollary.

Corollary 3.14. *Let T be an odomutant built from the odometer S on $X = \prod_{n \geq 0} \{0, \dots, q_n - 1\}$. If T is conjugate to an odometer, then T is conjugate to S .* \square

Proof of Theorem 3.13. Since T factors onto S , we already know that $\text{Sp}(S) \subset \text{Sp}(T)$. Let λ be an eigenvalue of T . Let us show that this is an eigenvalue of S . Let $\varepsilon > 0$ small enough so that $\theta < 1/4$ and $\frac{3\theta}{1-2\theta} \leq 1/4$, with $\theta = \theta(\varepsilon)$ introduced in Lemma 3.12. We also assume that $\varepsilon \leq 1/2$. Let n be a positive integer given by Lemma 3.11 for the eigenvalue λ , and $\nu := \lambda^{h_n}$. If $\nu = 1$, we are done.

Now assume $\nu \neq 1$. Let us choose a sufficiently large enough integer m so that $m \geq n$ and $\frac{6\theta}{\tau q_n \dots q_m} \leq \frac{1}{4}$. We consider a set $E_{n,m} \subset \prod_{j=n}^m \{0, \dots, q_j - 1\}$ and an integer $x_{m+1} \in \{0, \dots, q_{m+1} - 1\}$ satisfying

- $\frac{|E_{n,m}|}{q_n q_{n+1} \dots q_m} > 1 - \varepsilon$;
- for every $y = (y_n, \dots, y_m), z = (z_n, \dots, z_m) \in E_{n,m}$, we have

$$|1 - \nu^{H(y) - H(z)}| < \varepsilon,$$

where $H: \prod_{j=n}^m \{0, \dots, q_j - 1\} \rightarrow \{0, \dots, q_n \dots q_m - 1\}$ is defined by

$$H: y = (y_n, \dots, y_m) \mapsto \sum_{j=n}^m \frac{h_j}{h_n} \sigma_{y_{j+1}}^{(j)}(y_j) \text{ with } y_{m+1} := x_{m+1}.$$

The existence of $E_{n,m}$ and x_m is granted by Lemma 3.11. Since $\varepsilon \leq 1/2$ and H is a bijection, there exists two different elements y and z in $E_{n,m}$ such that $H(y) - H(z) = 1$. This implies

$$|1 - \nu| < \varepsilon.$$

Let us fix $z \in E_{n,m}$ and set

$$J = \left\{ H(y) - H(z) \mid y \in \prod_{j=n}^m \{0, \dots, q_j - 1\} \right\}.$$

By Lemma 3.12, we have

$$\sum_{j \in J} \mathbb{1}_{|1-\nu^j| < \varepsilon} \leq \frac{3\theta}{1-2\theta} |J| + \frac{6\theta}{|\tau|} \leq \frac{q_n \dots q_m}{2}$$

and we get a contradiction since we have

$$\sum_{j \in J} \mathbb{1}_{|1-\nu^j| < \varepsilon} \geq |E_{n,m}| > (1-\varepsilon)q_n \dots q_m$$

with $\varepsilon \leq 1/2$. Thus we have $\lambda^{h_n} = 1$. □

3.4 Orbit equivalence between odometers and odomutants

In this section, we prove that an odomutant and its associated odometer have the same orbits. Moreover, given a non-decreasing map $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, we give sufficient conditions for the cocycles to be φ -integrable.

Proposition 3.15. *For all $x \in \psi^{-1}(X_\infty^+)$, we have $Tx = S^{c_T(x)}x$ where the integer $c_T(x)$ is defined by*

$$c_T(x) = \sum_{i=0}^{N_1} h_i(y_i(x) - x_i) \tag{4}$$

with $N_1 := N^+(\psi(x))$ and $y_0, \dots, y_{N_1}(x)$ inductively defined by

$$\begin{aligned} y_{N_1}(x) &:= \left(\sigma_{x_{N_1+1}}^{(N_1)} \right)^{-1} (\sigma_{x_{N_1+1}}^{(N_1)}(x_{N_1}) + 1), \\ \forall 0 \leq i \leq N_1 - 1, y_i(x) &:= \left(\sigma_{y_{i+1}(x)}^{(i)} \right)^{-1} (0). \end{aligned}$$

For all $x \in X_\infty^+$, let us define the integer $c_S(x)$ by:

$$\begin{aligned} c_S(x) &= h_{N_2} \left(\sigma_{x_{N_2+1}}^{(N_2)}(1 + x_{N_2}) - \sigma_{x_{N_2+1}}^{(N_2)}(x_{N_2}) \right) \\ &\quad + h_{N_2-1} \left(\sigma_{1+x_{N_2}}^{(N_2-1)}(0) - \sigma_{x_{N_2}}^{(N_2-1)}(x_{N_2-1}) \right) \\ &\quad + \sum_{i=0}^{N_2-2} h_i \left(\sigma_0^{(i)}(0) - \sigma_{x_{i+1}}^{(i)}(x_i) \right) \end{aligned} \tag{5}$$

with $N_2 := N^+(x)$. Then we have $Sx = T^{c_S(x)}x$ for every $x \in X_\infty^+$.

Proof of Proposition 3.15. For $x \in \psi^{-1}(X_\infty^+)$, the value of $c_T(x)$ follows from the computations before Definition 3.2. For $x \in X_\infty^+$ and $N_2 := N^+(x)$, we have

$$x = (q_0 - 1, \dots, q_{N_2-1} - 1, \underbrace{x_{N_2}}_{\neq q_{N_2}-1}, x_{N_2+1}, x_{N_2+2}, \dots)$$

$$\text{and } Sx = (0, \dots, 0, 1 + x_{N_2}, x_{N_2+1}, x_{N_2+2}, \dots)$$

so the second result is clear by Corollary 3.9. □

Theorem 3.16. *The map $\Psi := id_X$ is an orbit equivalence between T and S . Moreover, given an non-decreasing map $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, this orbit equivalence is φ -integrable if one of the following two conditions is satisfied:*

(C1) *the series $\sum \frac{\varphi(h_{n+1})}{h_n}$ converges;*

(C2) *the series*

$$\sum_{n \geq 0} \frac{1}{h_{n+2}} \sum_{\substack{0 \leq x_n < q_n, \\ 0 \leq x_{n+1} < q_{n+1}, \\ \sigma_{x_{n+1}}^{(n)}(x_n) \neq q_n - 1}} \varphi \left(h_n \left(1 + \left| \left(\sigma_{x_{n+1}}^{(n)} \right)^{-1} (\sigma_{x_{n+1}}^{(n)}(x_n) + 1) - x_n \right| \right) \right)$$

$$\text{and } \sum_{n \geq 0} \frac{1}{h_{n+2}} \sum_{\substack{0 \leq x_n \leq q_n - 2, \\ 0 \leq x_{n+1} < q_{n+1}}} \varphi \left(h_n \left(1 + \left| \sigma_{x_{n+1}}^{(n)}(1 + x_n) - \sigma_{x_{n+1}}^{(n)}(x_n) \right| \right) \right)$$

converge.

As we notice in the next proof, we need coarse bounds to get that Condition (C1) implies φ -integrably orbit equivalence, whereas Condition (C2) is a finer hypothesis. For Theorem D, Condition (C2) will enable us to exploit the sublinearity of the map φ , and Condition (C1) will be enough for Theorems A and C.

Proof of Theorem 3.16. By Proposition 3.15, the set of points $x \in X$ satisfying $Tx = S^{c_T(x)}x$ and $Sx = T^{c_S(x)}x$ for integers $c_T(x)$ and $c_S(x)$ defined by (4) and (5) have full measure, so the map id_X is an orbit equivalence between S and T .

The value of $c_T(x)$ gives the following bound:

$$|c_T(x)| \leq h_{N_1} \left| \left(\sigma_{x_{N_1+1}}^{(N_1)} \right)^{-1} (\sigma_{x_{N_1+1}}^{(N_1)}(x_{N_1}) + 1) - x_{N_1} \right| + \underbrace{\sum_{i=0}^{N_1-1} h_i |y_i(x) - x_i|}_{\leq h_{N_1}} \quad (6)$$

with $N_1 = N^+(\psi(x))$. Given $n \geq 0$, $z_n \in \{0, \dots, q_n - 1\}$ and $z_{n+1} \in \{0, \dots, q_{n+1} - 1\}$ such that $\sigma_{z_{n+1}}^{(n)}(z_n) \neq q_n - 1$, we have

$$\mu(\{x \in X \mid N^+(\psi(x)) = n, x_n = z_n, x_{n+1} = z_{n+1}\}) = \frac{1}{h_{n+2}}.$$

We finally get

$$\begin{aligned} \int_X \varphi(|c_T(x)|) d\mu(x) &= \sum_{n \geq 0} \sum_{\substack{0 \leq z_n < q_n, \\ 0 \leq z_{n+1} < q_{n+1}, \\ \sigma_{z_{n+1}}^{(n)}(z_n) \neq q_n - 1}} \int_{\substack{N^+(\psi(x))=n, \\ x_n=z_n, \\ x_{n+1}=z_{n+1}}} \varphi(|c_T(x)|) d\mu(x) \\ &\leq \sum_{n \geq 0} \frac{1}{h_{n+2}} \sum_{\substack{0 \leq z_n < q_n, \\ 0 \leq z_{n+1} < q_{n+1}, \\ \sigma_{z_{n+1}}^{(n)}(z_n) \neq q_n - 1}} \varphi \left(h_n \left(1 + \left| \left(\sigma_{z_{n+1}}^{(n)} \right)^{-1} (\sigma_{z_{n+1}}^{(n)}(z_n) + 1) - z_n \right| \right) \right). \end{aligned}$$

From Inequality (6), we also get $|c_T(x)| \leq h_{N_1+1}$ and the following coarser bound:

$$\begin{aligned}
\int_X \varphi(|c_T(x)|) d\mu(x) &= \sum_{n \geq 0} \sum_{\substack{0 \leq z_n < q_n, \\ 0 \leq z_{n+1} < q_{n+1}, \\ \sigma_{z_{n+1}}^{(n)}(z_n) \neq q_n - 1}} \int_{\substack{N^+(\psi(x))=n, \\ x_n=z_n, \\ x_{n+1}=z_{n+1}}} \varphi(|c_T(x)|) d\mu(x) \\
&\leq \sum_{n \geq 0} \frac{1}{h_{n+2}} \sum_{\substack{0 \leq z_n < q_n, \\ 0 \leq z_{n+1} < q_{n+1}, \\ \sigma_{z_{n+1}}^{(n)}(z_n) \neq q_n - 1}} \varphi(h_{n+1}) \\
&\leq \sum_{n \geq 0} \frac{1}{h_n} \varphi(h_{n+1}).
\end{aligned}$$

For the other cocycle, we have

$$\begin{aligned}
|c_S(x)| &\leq h_{N_2} \left| \sigma_{x_{N_2+1}}^{(N_2)}(1 + x_{N_2}) - \sigma_{x_{N_2+1}}^{(N_2)}(x_{N_2}) \right| \\
&\quad + \underbrace{h_{N_2-1} \left| \sigma_{1+x_{N_2}}^{(N_2-1)}(0) - \sigma_{x_{N_2}}^{(N_2-1)}(x_{N_2-1}) \right| + \sum_{i=0}^{N_2-2} h_i \left| \sigma_0^{(i)}(0) - \sigma_{x_{i+1}}^{(i)}(x_i) \right|}_{\leq h_{N_2}}.
\end{aligned}$$

with $N_2 = N^+(x)$. Moreover it is easy to get

$$\mu(\{x \in X \mid N^+(x) = n, x_n = z_n, x_{n+1} = z_{n+1}\}) = \frac{1}{h_{n+2}}$$

for every $n \geq 0$, $z_n \in \{0, \dots, q_n - 2\}$ and $z_{n+1} \in \{0, \dots, q_{n+1} - 1\}$. Thus we find a bound on the φ -integral of c_S with the same method as c_T . \square

3.5 Extension to a homeomorphism on the Cantor set, strong orbit equivalence

We move on to a topological viewpoint. We give a sufficient condition for an odomutant to have an extension to a homeomorphism. It turns out that in this case the orbit equivalence that we obtained in the last section is a strong orbit equivalence.

Proposition 3.17. *Assume that $\sigma_i^{(n)}(0) = 0$ and $\sigma_i^{(n)}(q_n - 1) = q_n - 1$ for every $n \geq 0$ and every $0 \leq i \leq q_n - 1$. Then the odomutant T admits a unique extension, also denoted by T , which is a homeomorphism on the whole compact set $X = \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\}$. It is furthermore strongly orbit equivalent to the associated odometer S . In particular, it follows from Proposition 2.20 that T is uniquely ergodic.*

Remark 3.18. In this case, the equality $S \circ \psi(x) = \psi \circ T(x)$ holds for all $x \in X$.

Proof of Proposition 3.17. Since, for every $n \geq 0$, the points 0 and $q_n - 1$ are fixed by the n -th permutations, x^- is the only point $x \in X$ satisfying $\psi(x) = x^-$ and x^+ is the only point $x \in X$ satisfying $\psi(x) = x^+$. This implies that we have

$$\psi^{-1}(X_\infty^-) = X_\infty^- = X \setminus \{x^-\} \text{ and } \psi^{-1}(X_\infty^+) = X_\infty^+ = X \setminus \{x^+\},$$

and T is a bijection from $X \setminus \{x^+\}$ to $X \setminus \{x^-\}$, so we set $Tx^+ := x^-$. The map $T: X \rightarrow X$ is now a well-defined bijection.

The odometer S and the maps ψ_n are continuous on X so it is not difficult to see that T is continuous on each point of $X \setminus \{x^+\}$. It is easy to check the equality $T([q_0 - 1, \dots, q_n - 1]_{n+1}) = [0, \dots, 0]_{n+1}$, so the continuity at x^+ is clear. Therefore $T: X \rightarrow X$ is continuous and invertible, where X is a Hausdorff compact space, so T is a homeomorphism.

By Proposition 3.15, we have $Tx = S^{c_T(x)}x$ and $Sx = T^{c_S(x)}$ for every $x \in X_\infty^+$, with $c_T(x)$ and $c_S(x)$ defined by (4) and (5). These relations are extended at x^+ , with $c_T(x^+) = c_S(x^+) = 1$. Thus S and T have the same orbits and it is clear that the cocycles are continuous on X_∞^+ (x^+ is the only point of discontinuity if the cocycles are not continuous).¹⁴ \square

4 On non-preservation of loose Bernoullicity property under $L^{<1/2}$ orbit equivalence

In this section, we prove that $L^{<1/2}$ orbit equivalence (in particular Shannon orbit equivalence) does not imply even Kakutani equivalence.

Theorem 4.1. *There exists an ergodic probability measure-preserving bijection T which is $L^{<1/2}$ orbit equivalent (in particular Shannon orbit equivalent) to the dyadic odometer but not evenly Kakutani equivalent to it.*

Feldman [Fel76] has built a zero-entropy system which is not loosely Bernoulli. In his construction, for some partition that we will specify, the elements in $[0, \dots, 0]_n$ produce words, describing the future, which are not pairwise f -close for the f -metric introduced in Section 2.4 (therefore, the underlying system is not loosely Bernoulli). The goal is to describe his system as an odomutant built from the dyadic odometer and permutations that we are going to define. These permutations will fix 0, so that we will be able to read the words produced by the points at the bottom of the towers (using Lemmas A.1 and A.3), with respect to the partition that we will consider.

Let us set $\tilde{q}_n := 2^{n+10}$, $q_n := (\tilde{q}_n)^{2\tilde{q}_{n+1}+3}$ and $c_n := \frac{q_n}{\tilde{q}_n} = (\tilde{q}_n)^{2\tilde{q}_{n+1}+2}$ for every $n \geq 0$, $h_0 := 1$ and $h_{n+1} := q_n h_n$. We inductively define words $a_i^{(n)}$ (we keep the notations of Feldman in his paper) for every $n \geq 0$ and every $i \in \{0, \dots, \tilde{q}_n - 1\}$. Let us start with \tilde{q}_0 different letters $a_0^{(0)}, \dots, a_{\tilde{q}_0-1}^{(0)}$ seen as words of length $h_0 = 1$. For $n \geq 0$, if words $a_0^{(n)}, \dots, a_{\tilde{q}_n-1}^{(n)}$ of length h_n have been set, then we define new words $a_0^{(n+1)}, \dots, a_{\tilde{q}_{n+1}-1}^{(n+1)}$, of length h_{n+1} , by

$$a_j^{(n+1)} = \left\langle \langle a_0^{(n)} \rangle^{(\tilde{q}_n)^{2(j+1)}} \langle a_1^{(n)} \rangle^{(\tilde{q}_n)^{2(j+1)}} \dots \langle a_{\tilde{q}_n-1}^{(n)} \rangle^{(\tilde{q}_n)^{2(j+1)}} \right\rangle^{(\tilde{q}_n)^{2(\tilde{q}_{n+1}-j)}},$$

where $\langle w \rangle^k$ denotes the concatenation of k copies of a word w .

For $j \in \{0, 1, \dots, \tilde{q}_{n+1}-1\}$, $\tau_j^{(n)}$ is a permutation of the set $\{0, 1, \dots, q_n-1\}$ which permutes the entries of the finite sequence

$$\mathbf{u} := (\underbrace{a_0^{(n)}, \dots, a_0^{(n)}}_{c_n \text{ times}}, \underbrace{a_1^{(n)}, \dots, a_1^{(n)}}_{c_n \text{ times}}, \dots, \underbrace{a_{\tilde{q}_n-1}^{(n)}, \dots, a_{\tilde{q}_n-1}^{(n)}}_{c_n \text{ times}})$$

so that the concatenation gives $a_j^{(n+1)}$, namely $\tau_j^{(n)}$ satisfies

$$a_j^{(n)} = u_{\tau_j^{(n)}(0)} \cdot u_{\tau_j^{(n)}(1)} \cdot \dots \cdot u_{\tau_j^{(n)}(q_n-1)}.$$

¹⁴We can notice that we have $\{T^n x^- \mid n \in \mathbb{N}\} = \{S^n x^- \mid n \in \mathbb{N}\}$ and $\{T^{-n} x^+ \mid n \in \mathbb{N}\} = \{S^{-n} x^+ \mid n \in \mathbb{N}\}$.

We now consider the odomutant T associated to the odometer S on the space $X = \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\}$, and built with uniformly \mathbf{c} -multiple permutations $\tau_j^{(n)}$ where $\mathbf{c} := (c_n, \tilde{q}_n)_{n \geq 0}$.

In view of the cutting-and-stacking construction behind the definition of this odomutant, we can be convinced that T is isomorphic to the non Bernoulli system built by Feldman. However we give more details on the fact that T is not loosely Bernoulli, based on the justifications of Feldman. Given $n \geq 0$, Lemmas A.1 and A.3 in Appendix A imply that the words $[\tilde{\mathcal{P}}(1)]_{h_n}(x)$ for $x \in [0, \dots, 0]_n$ (i.e. the points x at the bottom of the towers at step n) exactly correspond to the words $a_0^{(n)}, \dots, a_{\tilde{q}_n-1}^{(n)}$. As in [Fel76], the properties we are interested in can be deduced purely from this fact. Indeed, given any point x not necessarily at the bottom of the towers at step n , the word $[\tilde{\mathcal{P}}(1)]_{h_n}(x)$ is the concatenation of the tail of some $a_i^{(n)}$ and the beginning of some $a_j^{(n)}$, and this observation leads us to apply the same reasoning as in [Fel76, Step III and Step V in p. 36] to conclude that T has zero entropy and $(T, \tilde{\mathcal{P}}(1))$ is not loosely Bernoulli using the characterisation provided by Theorem 2.8. Therefore T is not loosely Bernoulli.

Proof of Theorem A. Let S be the odometer on $X = \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\}$ (so S is loosely Bernoulli) and T the odomutant described above, which is not loosely Bernoulli, so that S and T are not evenly Kakutani equivalent by the theory of Ornstein, Rudolph and Weiss (see Theorem 2.9). Note that S is the dyadic odometer since the integers q_n are powers of 2.

Let us prove that S and T are $L^{<1/2}$ orbit equivalent, using Condition (C1) of Theorem 3.16. Let us fix a real number p satisfying $0 < p < 1/2$. We have

$$h_n = h_{n-1} \tilde{q}_{n-1}^{2\tilde{q}_n+3} = h_{n-1} 2^{(n+9)(2^{n+11}+3)}$$

and this gives $h_n = 2^{S_n}$ with

$$S_n := \sum_{i=1}^n (i+9)(2^{i+11}+3) = Cn2^n + D2^n + o(2^n)$$

for some positive constants C and D . For a fixed constant $C' \in [C, \frac{C}{2p}]$ and for a sufficiently large integer n , we have

$$Cn2^n < S_n < C'n2^n,$$

this gives

$$h_n < 2^{C'n2^n} = \left(2^{C(n-1)2^{n-1}}\right)^{2\frac{C'}{C}} 2^{C'2^n} < (2^{S_{n-1}})^{2\frac{C'}{C}} 2^{C'2^n} = (h_{n-1})^{2\frac{C'}{C}} 2^{C'2^n}$$

and

$$\frac{h_n^p}{h_{n-1}} < (h_{n-1})^{2\frac{C'}{C}p-1} 2^{C'p2^n} < \left(2^{C'(n-1)2^{n-1}}\right)^{2\frac{C'}{C}p-1} 2^{C'p2^n}.$$

Since we have $2\frac{C'}{C}p < 1$, the series $\sum \frac{\varphi(h_n)}{h_{n-1}}$ converges for $\varphi(x) = x^p$, so we are done by Theorem 3.16. \square

5 On non-preservation of entropy under orbit equivalence with almost log-integrable cocycles

In this section, we prove that orbit equivalence with almost log-integrable cocycles does not preserve entropy. The statement is actually stronger, with a topological framework:

Theorem 5.1. *Let α be either a positive real number or $+\infty$. Let S be an odometer whose associated supernatural number $\prod_{p \in \Pi} p^{k_p}$ satisfies the following property: there exists a prime number p_* such that $k_{p_*} = +\infty$. Then there exists a Cantor minimal homeomorphism T such that*

1. $h_{\text{top}}(T) = \alpha$;
2. *there exists a strong orbit equivalence between S and T , which is φ_m -integrable for all integers $m \geq 0$,*

where φ_m denotes the map $t \rightarrow \frac{\log t}{\log^{(\circ m)} t}$ and $\log^{(\circ m)}$ the composition $\log \circ \dots \circ \log$ (m times).

We will crucially use the combinatorial lemmas stated in Appendix A. The cases $\alpha < +\infty$ and $\alpha = +\infty$ will be in fact separated, but in both proofs, we will apply the following lemma which will be useful for the quantification of the cocycles.

Lemma 5.2. *Let $(q_n)_{n \geq 0}$ be a sequence of integers greater than or equal to 2, and let $\beta > 0$ such that*

$$\liminf_{n \rightarrow +\infty} \frac{\log q_n}{h_n} \geq \beta$$

where $h_n := q_0 \dots q_{n-1}$. Then, for every integer $m \geq 0$, we have

$$\frac{1}{\log^{(\circ m)}(q_{n+m})} \leq \exp(-\beta h_n)$$

for all sufficiently large integers n . In particular, the sequence $\left(\frac{1}{\log^{(\circ m)}(q_n)}\right)_{n \geq 0}$ is summable.

Proof of Lemma 5.2. Let us consider an integer N such that

$$\forall n \geq N, \beta h_n \geq 1.$$

For every $n \geq N$, we have

$$\log q_{n+1} \geq \beta h_{n+1} = q_n \times \beta h_n \geq q_n.$$

By induction, we easily get for every $n \geq N$,

$$\log^{(\circ m)}(q_{n+m}) \geq q_n,$$

so we have $\log^{(\circ m)}(q_{n+m}) \geq \exp(\beta h_n)$. □

Before the proof of Theorem 5.1 in the case $\alpha < +\infty$, we need two preliminary lemmas. The first one (Lemma 5.3) provides permutations so that we can easily compute the entropy of the underlying odomutant. The second one (Lemma 5.5) proves that the formula given by Lemma 5.3 enables us to get all possible finite values of the entropy with a proper choice of parameters.

Lemma 5.3. *Let $(q_n)_{n \geq 0}$ be a sequence of integers greater than or equal to 2 and satisfying $q_{n+1} \leq (q_n - 2)!$. Then there exist permutations $\sigma_{x_{n+1}}^{(n)}$, for $n \geq 0$ and $x_{n+1} \in \{0, \dots, q_{n+1} - 1\}$, satisfying the following properties:*

1. *for every $n \geq 0$, the maps $\sigma_0^{(n)}, \sigma_1^{(n)}, \dots, \sigma_{q_{n+1}-1}^{(n)}$ are pairwise different permutations of the set $\{0, \dots, q_n - 1\}$, fixing 0 and $q_n - 1$;*

2. the topological entropy of the underlying odomutant is equal to

$$\lim_{n \rightarrow +\infty} \frac{\log q_n}{h_n}.$$

Remark 5.4. The entropy $h_{\text{top}}(T)$ is well-defined by Lemma 3.17, as well as the limit $\lim_{n \rightarrow +\infty} \frac{\log q_n}{h_n}$. Indeed, the sequence $\left(\frac{\log q_n}{h_n}\right)_{n \geq 0}$ is decreasing since we have $q_n < (q_{n-1})^{q_{n-1}}$.

Proof of Lemma 5.3. By Lemma A.6 in Appendix A, we can choose permutations satisfying the first item and such that the following hold for every $n \geq \ell \geq 1$:

$$q_n \leq N(\mathcal{P}(\ell)_0^{h_n-1}) \leq h_{n-1} q_n q_{n-1}^2 2^{q_{n-1}}.$$

On the one hand, we have

$$h_{\text{top}}(T) \geq h_{\text{top}}(T, \mathcal{P}(1)) = \lim_{n \rightarrow +\infty} \frac{\log N(\mathcal{P}(1)_0^{h_n-1})}{h_n} \geq \lim_{n \rightarrow +\infty} \frac{\log q_n}{h_n}.$$

On the other hand, this gives

$$\begin{aligned} h_{\text{top}}(T, \mathcal{P}(\ell)) &= \lim_{n \rightarrow +\infty} \frac{\log N(\mathcal{P}(\ell)_0^{h_n-1})}{h_n} \\ &\leq \lim_{n \rightarrow +\infty} \frac{\log h_{n-1} + \log q_n + 2 \log q_{n-1} + q_{n-1} \log 2}{h_n} \\ &= \lim_{n \rightarrow +\infty} \frac{\log q_n}{h_n} \end{aligned}$$

and we finally get, using Theorem 2.2,

$$h_{\text{top}}(T) = \lim_{\ell \rightarrow +\infty} h_{\text{top}}(T, \mathcal{P}(\ell)) \leq \lim_{n \rightarrow +\infty} \frac{\log q_n}{h_n}.$$

Hence the result. \square

Lemma 5.5. Let α be a positive real number and $\prod_{p \in \Pi} p^{k_p}$ a supernatural number. We assume that there exists a prime number p_\star such that $k_{p_\star} = +\infty$. Then there exists a sequence $(q_n)_{n \geq 0}$ of integers greater than or equal to 2, satisfying the following properties:

1. we have $q_{n+1} \leq (q_n - 2)!$ for every $n \geq 0$;

2. the sequence $\left(\frac{\log q_n}{q_0 \dots q_{n-1}}\right)_{n \geq 0}$ tends to α ;

3. we have $\sum_{n \geq 0} \nu_p(q_n) = k_p$ for every $p \in \Pi$.

Proof of Lemma 5.5. Let K be a large enough power of p_\star so that the following property holds: for every integer q satisfying $q \geq K$, we have $(q - 2)! \geq K$. Let

$$N := \sum_{p \in \Pi \setminus \{p_\star\}} k_p \in \mathbb{N} \cup \{+\infty\}$$

and $(p_i)_{1 \leq i \leq N}$ be a sequence of prime numbers satisfying $\sum_{1 \leq i \leq N} \mathbb{1}_{p_i=p} = k_p$ for every $p \in \Pi \setminus \{p_\star\}$, and $\sum_{1 \leq i \leq N} \mathbb{1}_{p_i=p_\star} = 0$.¹⁵ By induction, we build a sequence $(q_n)_{n \geq 0}$ of integers greater than or equal to 2, an increasing sequence $(i_n)_{n \geq 0}$ and a non-decreasing sequence $(j_n)_{n \geq 0}$ of non-negative integers, satisfying the following properties:

¹⁵" $(p_i)_{i \geq 1}$ " and " $\sum_{i \geq 1}$ " in the case $N = +\infty$.

1. $q_0 > \frac{2}{\alpha} \log p_\star$ and $\log q_0 \geq \alpha + 5$;
2. $K \leq q_{n+1} \leq (q_n - 2)!$ for every $n \geq 0$;
3. for every $n \geq 1$, the following holds:

$$\alpha + \frac{5}{q_0 \dots q_{n-1}} \leq \frac{\log q_n}{q_0 \dots q_{n-1}} \leq \alpha + \frac{2}{q_0 \dots q_{n-2}},$$

where $q_0 \dots q_{n-2}$ is equal to 1 if $n = 1$;

4. $q_n = K^{i_n} \prod_{j=j_{n-1}+1}^{j_n} p_j$ for every $n \geq 0$, with $j_{-1} = j_0 = 0$ (so we have $q_0 = p_\star^{i_0}$);
5. $j_n \xrightarrow{n \rightarrow \infty} N$ if $N < +\infty$, $j_n \xrightarrow{n \rightarrow \infty} +\infty$ otherwise.

Such a sequence $(q_n)_{n \geq 0}$ satisfies the assumptions of the lemma.

We choose a large enough integer i_0 such that the hypotheses on $q_0 := K^{i_0}$ are satisfied. Let $n \geq 0$. Assume that the integers $q_0, \dots, q_n, i_0, \dots, i_n, j_1, \dots, j_n$ have been defined and let us build $q_{n+1}, i_{n+1}, j_{n+1}$. In particular, the integers q_0, \dots, q_n satisfy

$$\forall k \in \{0, \dots, n\}, \frac{\log q_k}{q_0 \dots q_{k-1}} \geq \alpha + \frac{5}{q_0 \dots q_{k-1}}.$$

Let j_{n+1} be the greatest integer k satisfying

- $k \geq j_n$ and, if $N < +\infty$, $k \leq N$;
- $K \prod_{j=j_n+1}^k p_j \leq (q_n - 2)!$;
- $\frac{\log \left(\prod_{j=j_n+1}^k p_j \right)}{q_0 \dots q_n} \leq \frac{\alpha}{2}$.

Let us consider the sequence $(\alpha_i)_{i \geq 1}$ defined by

$$\alpha_i := \frac{\log \left(K^i \prod_{j=j_n+1}^{j_{n+1}} p_j \right)}{q_0 \dots q_n},$$

and let I be the greatest integer i such that $K^i \prod_{j=j_n+1}^{j_{n+1}} p_j \leq (q_n - 2)!$. The sequence $(\alpha_i)_{i \geq 1}$ is an arithmetic progression with common difference

$$\frac{\log K}{q_0 \dots q_n}.$$

Moreover, we have

$$\alpha_1 \leq \frac{\log K}{q_0} + \frac{\log \left(\prod_{j=j_n+1}^{j_{n+1}} p_j \right)}{q_0 \dots q_n} \leq \alpha,$$

and, using the assumption on q_n and the inequalities $\log(k!) \geq k \log(k) - k$ and $\log k \leq k$,

$$\begin{aligned}
\alpha_{I+1} &= \frac{\log \left(K^{I+1} \prod_{j=j_n+1}^{j_{n+1}} p_j \right)}{q_0 \dots q_n} \\
&\geq \frac{\log((q_n - 2)!)}{q_0 \dots q_n} \\
&= \frac{\log(q_n!)}{q_0 \dots q_n} - \frac{\log(q_n - 1)}{q_0 \dots q_n} - \frac{\log(q_n)}{q_0 \dots q_n} \\
&\geq \frac{q_n \log(q_n) - q_n}{q_0 \dots q_n} - \frac{\log(q_n - 1)}{q_0 \dots q_n} - \frac{\log(q_n)}{q_0 \dots q_n} \\
&\geq \frac{\log(q_n)}{q_0 \dots q_{n-1}} - \frac{3}{q_0 \dots q_{n-1}} \\
&\geq \alpha + \frac{2}{q_0 \dots q_{n-1}}.
\end{aligned}$$

Therefore, there exists $i' \in \{1, \dots, I\}$ such that

$$\alpha + \frac{2}{q_0 \dots q_{n-1}} - \frac{\log K}{q_0 \dots q_n} \leq \alpha_{i'} \leq \alpha + \frac{2}{q_0 \dots q_{n-1}}.$$

Since we have $2q_n \geq 2K \geq K + \log K \geq 5 + \log K$, we get

$$\alpha + \frac{5}{q_0 \dots q_n} \leq \alpha_{i'} \leq \alpha + \frac{2}{q_0 \dots q_{n-1}}.$$

It remains to set $i_{n+1} := i'$ and $q_{n+1} := K^{i_{n+1}} \prod_{j=j_n+1}^{j_{n+1}} p_j$.

Finally, we have to check that the increasing sequence $(j_n)_{n \geq 1}$ of integers diverges if $N = +\infty$, or converges to N if N is finite. If it was not the case, then there would exist a positive integer n such that the following hold for every $k \geq n$:

$$K p_{j_n+1} > (q_k - 2)! \text{ or } \frac{\log p_{j_n+1}}{q_0 \dots q_k} > \frac{\alpha}{2}.$$

But the integers q_k are greater than or equal to 2, so it would mean that the sequence $(q_k)_{k \geq 0}$ is bounded, which is in contradiction with the inequality $\log q_k \geq \alpha q_0 \dots q_k$, so $(j_n)_{n \geq 1}$ satisfies the desired property. Hence the lemma. \square

Proof of Theorem C in the case $\alpha < +\infty$. Let α be a positive real number and let S be an odometer whose associated supernatural number $\prod_{p \in \Pi} p^{k_p}$ satisfies the following property: there exists a prime number p_\star such that $k_{p_\star} = +\infty$. Without loss of generality, S is the odometer on the Cantor set $X := \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\}$, where the sequence $(q_n)_{n \geq 0}$ satisfies $2 \leq q_n \leq (q_n - 2)!$ for every $n \geq 0$ and $\frac{\log q_n}{h_n} \rightarrow \alpha$. The existence of such a sequence is granted by Lemma 5.5. By Lemma 5.3 and Proposition 3.17, we can find families of permutations such that the underlying odomutant T is a homeomorphism strongly orbit equivalent to S and its topological entropy is equal to α .

Finally, given an increasing map $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, the orbit equivalence is φ -integrable if $(\varphi(h_{n+1})/h_n)_n$ is summable, by Theorem 3.16 (see Condition (C1)). This holds for $\varphi(x) =$

$\frac{\log(x)}{\log^{\circ m}(x)}$. Indeed, we have

$$\begin{aligned} \frac{\varphi(h_{n+1})}{h_n} &= \frac{1}{\log^{\circ m}(h_{n+1})} \frac{\log(h_{n+1})}{h_n} \\ &\leq \frac{1}{\log^{\circ m}(q_n)} \left(\frac{\log(h_n)}{h_n} + \frac{\log q_n}{h_n} \right) \\ &\leq \frac{1}{\log^{\circ m}(q_n)} \left(1 + \frac{\log q_n}{h_n} \right), \end{aligned}$$

so using the monotonicity of the sequence $\left(\frac{\log q_n}{h_n}\right)_{n \geq 0}$ (see Remark 5.4), we get

$$\frac{\varphi(h_{n+1})}{h_n} \leq \frac{1}{\log^{\circ m}(q_n)} (1 + \log q_0)$$

and we are done by Lemma 5.2 with $\beta = \alpha$. \square

In the case $\alpha = +\infty$, we prove Theorem C with the same methods as in [BH94], but with our formalism. We will consider an odomutant T on $\prod_{n \geq 0} \{0, 1, \dots, q_n - 1\}$, built with uniform \mathbf{c} -multiple permutations $\tau_j^{(n)}$, where $\mathbf{c} = (c_n, \tilde{q}_n)_{n \geq 0}$, and for every $n \geq 0$ and every $0 \leq j < \tilde{q}_{n+1}$, $\tau_j^{(n)}$ is a permutation on $\{0, 1, \dots, q_n - 1\}$ fixing 0 and $q_n - 1$. For every $n \geq 0$, we assume that the map

$$j \in \{0, 1, \dots, \tilde{q}_{n+1} - 1\} \mapsto (\tau_j^{(n)}(I_0^{(n)}), \dots, \tau_j^{(n)}(I_{\tilde{q}_n-1}^{(n)}))$$

is κ_{n+1} -to-one for some positive integer κ_{n+1} (as in the assumption of Lemma A.4). Finally, we write $\chi_n := \frac{\tilde{q}_n}{\kappa_n}$ for every $n \geq 1$. Then we have $q_n = c_n \tilde{q}_n$ for every $n \geq 0$ and $\tilde{q}_n = \kappa_n \chi_n$ for every $n \geq 1$. The sequences (h_n) , (\tilde{q}_n) , (c_n) , (κ_n) , (χ_n) respectively correspond to the sequences (l_k) , (m_k) , (n_k) , (j_k) , (\bar{m}_k) in [BH94]. The integer χ_{n+1} is the number of sequences of the form $(\tau_j^{(n)}(I_0^{(n)}), \dots, \tau_j^{(n)}(I_{\tilde{q}_n-1}^{(n)}))$ for $j \in \{0, \dots, \tilde{q}_{n+1} - 1\}$, so we have

$$1 \leq \chi_{n+1} \leq \frac{(q_n - 2)!}{c_n! \tilde{q}_n - 2 (c_n - 1)!^2},$$

thus motivating the following lemma.

Lemma 5.6. *Let p , \tilde{q} and c be positive integers and $q := \tilde{q}c$. Assume that $p \geq 2$ and $q \geq 3$. Then the greatest power of p less than or equal to*

$$\frac{(q - 2)!}{c! \tilde{q} - 2 (c - 1)!^2}$$

is greater than or equal to

$$\frac{1}{p} \frac{1}{\tilde{q}^2} \left(\frac{1}{ec} \right)^{\tilde{q}} \tilde{q}^q.$$

Proof of Lemma 5.6. Using the inequalities

$$\left(\frac{k}{e} \right)^k \leq k! \leq e \left(\frac{k+1}{e} \right)^{k+1},$$

we get

$$\frac{(q-2)!}{c!^{\tilde{q}-2}(c-1)!^2} = \frac{c^2}{q(q-1)} \frac{q!}{(c-1)!^{\tilde{q}}} \frac{1}{c^{\tilde{q}}} \geq \left(\frac{c}{q}\right)^2 \frac{\left(\frac{q}{c}\right)^q}{e^{\tilde{q}} \left(\frac{c}{e}\right)^{c\tilde{q}}} \frac{1}{c^{\tilde{q}}} = \frac{1}{\tilde{q}^2} \left(\frac{1}{ec}\right)^{\tilde{q}} \tilde{q}^q$$

and we are done. \square

Proof of Theorem C in the case $\alpha = +\infty$. Let

$$N := \sum_{p \in \Pi \setminus \{p_\star\}} k_p \in \mathbb{N} \cup \{+\infty\}$$

and $(p_i)_{1 \leq i \leq N}$ be a sequence of prime numbers satisfying $\sum_{1 \leq i \leq N} \mathbb{1}_{p_i=p} = k_p$ for every $p \in \Pi \setminus \{p_\star\}$, and $\sum_{1 \leq i \leq N} \mathbb{1}_{p_i=p_\star} = 0$.¹⁶ Let us define $c_n = p_\star^n$ for every $n \geq 0$. By induction, we build sequences $(\kappa_n)_{n \geq 1}$ and $(\chi_n)_{n \geq 0}$ of integers, and a non-decreasing sequence $(j_n)_{n \geq 1}$ of non-negative integers, satisfying the following properties:

1. for every $n \geq 0$, χ_{n+1} is the greatest power of p_\star less than or equal to $\frac{(q_n-2)!}{c_n!^{\tilde{q}_n-2}(c_n-1)!^2}$, where $\tilde{q}_n = \kappa_n \chi_n$ (with $\kappa_0 := 1$) and $q_n := c_n \tilde{q}_n$;
2. $\kappa_n = p_\star^{h_n} \prod_{j=j_{n-1}+1}^{j_n} p_j$ for every $n \geq 1$, with $j_0 := 0$;
3. $j_n \xrightarrow{n \rightarrow \infty} N$ if $N < +\infty$, $j_n \xrightarrow{n \rightarrow \infty} +\infty$ otherwise.

Let us define $\tilde{q}_0 := p_\star$. Given $n \geq 0$, assume that $\chi_0, \dots, \chi_n, j_0, \dots, j_n, \kappa_1, \dots, \kappa_n$ have been set (if $n = 0$, then there is no integer κ_i). We define χ_{n+1} as the greatest power of p_\star less than or equal to $\frac{(q_n-2)!}{c_n!^{\tilde{q}_n-2}(c_n-1)!^2}$, j_{n+1} as the greatest integer k satisfying

- $j_n \leq k$ and, if $N < +\infty$, $k \leq N$;
- $\prod_{j=j_{n+1}}^k p_j \leq p_\star^{h_{n+1}}$,

and $\kappa_{n+1} := p_\star^{h_{n+1}} \prod_{j=j_{n+1}}^{j_{n+1}} p_j$. Let us define T as the odomutant built with uniform \mathbf{c} -multiple permutations $\tau_j^{(n)}$, with $\mathbf{c} := (c_n, \tilde{q}_n)_{n \geq 0}$, and assume that the assumption of Lemma A.4 in Appendix A is satisfied: for every $n \geq 0$, the map

$$j \in \{0, 1, \dots, \tilde{q}_{n+1} - 1\} \mapsto (\tau_j^{(n)}(I_0^{(n)}), \dots, \tau_j^{(n)}(I_{\tilde{q}_n-1}^{(n)}))$$

is κ_{n+1} -to-1. Note that the fact that χ_{n+1} is less than or equal to

$$\frac{(q_{n+1}-2)!}{c_{n+1}!^{\tilde{q}_{n+1}-2}(c_{n+1}-1)!^2}$$

enables us to find such families of permutations. It is straightforward to prove that $j_n \rightarrow +\infty$ if $N = \infty$, or $j_n \rightarrow N$ if $N < +\infty$, so T is an odomutant associated to S .

Lemma A.4 implies

$$N(\tilde{\mathcal{P}}(\ell)_0^{h_n-1}) \geq \frac{\tilde{q}_n}{\prod_{k=\ell}^n \kappa_k^{h_n/h_k}}$$

¹⁶ " $(p_i)_{i \geq 1}$ " and " $\sum_{i \geq 1}$ " in the case $N = +\infty$.

for all $n \geq \ell \geq 1$. By Lemma 5.6, we have for every $i \geq 1$,

$$\tilde{q}_i = \kappa_i \chi_i \geq \kappa_i \frac{1}{p_\star} \frac{1}{(\tilde{q}_{i-1})^2} \left(\frac{1}{ec_{i-1}} \right)^{\tilde{q}_{i-1}} (\tilde{q}_{i-1})^{q_{i-1}},$$

this gives

$$\tilde{q}_i^{1/h_i} \geq \kappa_i^{1/h_i} \left(\frac{1}{p_\star \tilde{q}_{i-1}^2} \right)^{1/h_i} \left(\frac{1}{ec_{i-1}} \right)^{1/(c_{i-1}h_{i-1})} \tilde{q}_{i-1}^{1/h_{i-1}}$$

and we can apply this inequality many times to get

$$\begin{aligned} \tilde{q}_n^{1/h_n} &\geq \left(\prod_{i=1}^n \kappa_i^{1/h_i} \right) \left(\prod_{i=1}^n \left(\frac{1}{p_\star \tilde{q}_{i-1}^2} \right)^{1/h_i} \left(\frac{1}{ec_{i-1}} \right)^{1/(c_{i-1}h_{i-1})} \right) \tilde{q}_0 \\ &\geq \left(\prod_{i=\ell}^n \kappa_i^{1/h_i} \right) \left(\prod_{i=1}^n \left(\frac{1}{p_\star \tilde{q}_{i-1}^2} \right)^{1/h_i} \left(\frac{1}{ec_{i-1}} \right)^{1/(c_{i-1}h_{i-1})} \right) p_\star^{\ell-1} \tilde{q}_0. \end{aligned}$$

Hence we have,

$$\frac{\log N(\tilde{\mathcal{P}}(\ell)_0^{h_n-1})}{h_n} \geq (\ell-1) \log p_\star + \log \tilde{q}_0 - \sum_{i=1}^n \left(\frac{\log(p_\star \tilde{q}_{i-1}^2)}{h_i} + \frac{\log(ec_{i-1})}{c_{i-1}h_{i-1}} \right).$$

It is straightforward to check that the series $\sum_{i=1}^{+\infty} \left(\frac{\log(p_\star \tilde{q}_{i-1}^2)}{h_i} + \frac{\log(ec_{i-1})}{c_{i-1}h_{i-1}} \right)$ converges and we denote by V its value. We are now able to get that T has infinite topological entropy:

$$h_{\text{top}}(T) \geq \lim_{\ell \rightarrow +\infty} h_{\text{top}}(T, \tilde{\mathcal{P}}(\ell)) \geq \lim_{\ell \rightarrow +\infty} ((\ell-1) \log p_\star + \log \tilde{q}_0 - V) = +\infty.$$

Let us finally check Condition (C1) in Lemma 3.16 to prove that there exists a strong orbit equivalence between T and S , which is φ_m -integrable for every $m \geq 0$, where $\varphi_m(x) = \frac{\log(x)}{\log^{(om)}(x)}$. We first have $c_n \leq (p_\star)^{h_n}$, $\chi_n \leq (q_{n-1})^{q_{n-1}} \leq (h_n)^{q_{n-1}}$ and $\log \kappa_n \leq 2h_n \log p_\star$ by definition, so

$$\log h_{n+1} = \log h_n + \log c_n + \log \kappa_n + \log \chi_n \leq (1 + 3 \log p_\star) h_n + q_{n-1} \log h_n,$$

this implies

$$\frac{\log h_{n+1}}{h_n} \leq (1 + 3 \log p_\star) + \frac{\log h_n}{h_{n-1}}$$

and we get $\frac{\log h_{n+1}}{h_n} = O(n)$. Then, it remains to prove that the sequence $\left(\frac{n}{\log^{(om)}(h_{n+1})} \right)_{n \geq 0}$ is summable. This is a consequence of Lemma 5.2 with $\beta = \log p_\star$, since we have

$$\log q_n \geq \log \kappa_n \geq h_n \log p_\star$$

by definition of κ_n . So there exists a strong orbit equivalence between T and S , which is φ_m -integrable for every $m \geq 0$. \square

6 Orbit equivalence with almost integrable cocycles

In this section, we prove that being orbit equivalent to an odometer, with almost integrable cocycles, does not imply being flip-conjugate to it.

Theorem 6.1. *Let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a sublinear map and S an odometer. There exists a probability measure-preserving transformation T such that S and T are φ -integrably orbit equivalent but not flip-conjugate.*

For Theorems A and C, some invariants (loose Bernoullicity property, entropy) ensure that we build an odomutant T which is not flip-conjugate to the associated odometer S . For Theorem 6.1, we use the fact that every odometer is coalescent (see Theorem 2.14). Given a sublinear map $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, the goal is to find families of permutations $(\sigma_{x_{n+1}}^{(n)})_{0 \leq x_{n+1} < q_{n+1}}$, for $n \geq 0$, such that the factor map

$$\psi: x \in X \rightarrow (\sigma_{x_1}^{(0)}(x_0), \sigma_{x_2}^{(1)}(x_1), \sigma_{x_3}^{(2)}(x_2), \dots) \in X$$

from the associated odomutant T to S is not an isomorphism, with φ -integrable cocycles for the orbit equivalence between S and T .

Lemma 6.2. *Let $(q_n)_{n \geq 0}$ be a sequence of integers greater or equal to 2. For every $n \geq 0$, let $(\sigma_{x_{n+1}}^{(n)})_{0 \leq x_{n+1} < q_{n+1}}$ be a family of permutations of the set $\{0, 1, \dots, q_n - 1\}$, defined by:*

$$\forall x_{n+1} \in \{0, \dots, q_{n+1} - 1\}, \forall i \in \{0, \dots, q_n - 1\}, \sigma_{x_{n+1}}^{(n)}(i) = i + x_{n+1} \bmod q_n.$$

Assume that the infinite product $\prod_{n \geq 0} \left(1 - \frac{1}{q_n}\right)$ converges¹⁷. Then $\psi: x \in X \rightarrow (\sigma_{x_{n+1}}^{(n)}(x_n))_{n \geq 0} \in X$ is not injective almost everywhere.

Proof of Lemma 6.2. Let $Y_1 := \{x \in X \mid \forall n \geq 0, x_n \neq (q_n - 1)\mathbb{1}_{n \text{ is even}}\}$ and $Y_2 := \{x \in X \mid \forall n \geq 0, x_n \neq (q_n - 1)\mathbb{1}_{n \text{ is odd}}\}$. It is straightforward to check that

$$\mu(Y_1) = \mu(Y_2) = \prod_{n \geq 0} \left(1 - \frac{1}{q_n}\right) > 0.$$

Let $\theta: X \rightarrow X$ defined by:

$$\theta(x) := (x_n + (-1)^n \bmod q_n)_{n \geq 0}.$$

The map θ is in $\text{Aut}(X, \mu)$ since X can be seen as the compact group $\prod_{n \geq 0} \mathbb{Z}/q_n\mathbb{Z}$, with its Haar probability measure μ and θ as the translation by $((-1)^n)_{n \geq 0}$. Moreover, θ is a bijection from Y_1 to Y_2 and we have $\psi(\theta(x)) = \psi(x)$ for all $x \in Y_1$.

Let us prove by contradiction that ψ is not injective almost everywhere. Assume that ψ is injective on a measurable set X_0 of full measure. This hypothesis and the equality $\psi \circ \theta = \psi$ on Y_1 imply that the sets X_0 and $\theta(X_0 \cap Y_1)$ are disjoint. This finally gives

$$\mu((X_0)^c) \geq \mu(\theta(X_0 \cap Y_1)) = \mu(X_0 \cap Y_1) = \mu(Y_1) > 0$$

and we get a contradiction since $(X_0)^c$ has zero measure. □

¹⁷By definition, the infinite product $\prod_{n \geq 0} \left(1 - \frac{1}{q_n}\right)$ converges if the sequence $\left(\prod_{k=0}^n \left(1 - \frac{1}{q_k}\right)\right)_{n \geq 0}$ converges to a nonzero real number.

Before the proof of Theorem D, we use a lemma stated in [CJLMT23] and which enables us to reduce to the case where the sublinear map φ is non-decreasing (actually the statement is stronger but we only need the monotonicity).

Lemma 6.3 (Lemma 2.12 in [CJLMT23]). *Let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a sublinear function. Then there is a sublinear non-decreasing function $\tilde{\varphi}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\varphi(t) \leq \tilde{\varphi}(t)$ for all t large enough.*

Proof of Theorem D. Let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a sublinear map. If $\tilde{\varphi}$ is another sublinear map satisfying $\varphi(t) = O(\tilde{\varphi}(t))$, then $\tilde{\varphi}$ -integrability implies φ -integrability. Therefore, by Lemma 6.3, we assume without loss of generality that φ is non-decreasing.

Let $(q_n)_{n \geq 0}$ be a sequence of integers greater or equal to 2 and S the odometer on $X := \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\}$. The Halmos-von Neumann theorem implies that S is conjugate to the odometer on $\prod_{n \geq 0} \{0, 1, \dots, q_{i_n-1} \dots q_{i_n-1} - 1\}$ for any increasing sequence $(i_n)_{n \geq 0}$ satisfying $i_0 = 0$. Therefore, we can assume without loss of generality that the integers q_n are sufficiently large so that they satisfy the following properties:

1. $\prod_{n \geq 0} \left(1 - \frac{1}{q_n}\right)$ converges¹⁷;
2. the series $\sum \frac{\varphi(2h_n)}{h_n}$ converges.

Let T be the odomutant built from S and the same families $\left(\sigma_{x_{n+1}}^{(n)}\right)_{0 \leq x_{n+1} < q_{n+1}}$ as in Lemma 6.2.

By this lemma and Theorem 2.14, S and T are not conjugate. Since S is conjugate to its inverse S^{-1} (by the Halmos-von Neumann theorem), S and T are not flip-conjugate.

It remains to quantify the cocycles, using Condition (C2) of Theorem 3.16. Let $n \geq 0$ and $x_{n+1} \in \{0, \dots, q_{n+1} - 1\}$, and $i \in \{0, \dots, q_n - 1\}$ such that $x_{n+1} = i \bmod q_n$. For every $x \in \{0, \dots, q_n - 2\} \setminus \{q_n - i - 1\}$, we have

$$\left(\sigma_{x_{n+1}}^{(n)}\right)^{-1} (\sigma_{x_{n+1}}^{(n)}(x_n) + 1) - x_n = \sigma_{x_{n+1}}^{(n)}(1 + x_n) - \sigma_{x_{n+1}}^{(n)}(x_n) = 1.$$

For $x_n = q_n - 1$, we consider the following bounds:

$$\left| \left(\sigma_{x_{n+1}}^{(n)}\right)^{-1} (\sigma_{x_{n+1}}^{(n)}(x_n) + 1) - x_n \right| \leq q_n$$

$$\text{and } \left| \sigma_{x_{n+1}}^{(n)}(1 + x_n) - \sigma_{x_{n+1}}^{(n)}(x_n) \right| \leq q_n.$$

We finally get

$$\begin{aligned}
& \sum_{n \geq 0} \frac{1}{h_{n+2}} \sum_{\substack{0 \leq x_n < q_n, \\ 0 \leq x_{n+1} < q_{n+1}, \\ \sigma_{x_{n+1}}^{(n)}(x_n) \neq q_n - 1}} \varphi \left(h_n \left(1 + \left| \left(\sigma_{x_{n+1}}^{(n)} \right)^{-1} (\sigma_{x_{n+1}}^{(n)}(x_n) + 1) - x_n \right| \right) \right) \\
&= \sum_{n \geq 0} \frac{1}{h_{n+2}} \sum_{\substack{0 \leq x_n \leq q_n - 2, \\ 0 \leq x_{n+1} < q_{n+1}, \\ x_n \neq q_n - i - 1}} \varphi \left(h_n \left(1 + \left| \left(\sigma_{x_{n+1}}^{(n)} \right)^{-1} (\sigma_{x_{n+1}}^{(n)}(x_n) + 1) - x_n \right| \right) \right) \\
&\leq \sum_{n \geq 0} \frac{1}{h_{n+2}} \sum_{0 \leq x_{n+1} < q_{n+1}} ((q_n - 2)\varphi(2h_n) + \varphi(h_n(1 + q_n))) \\
&\leq \sum_{n \geq 0} \frac{\varphi(2h_n)}{h_n} + \sum_{n \geq 0} \frac{\varphi(2h_{n+1})}{h_{n+1}} < +\infty
\end{aligned}$$

and similarly

$$\sum_{n \geq 0} \frac{1}{h_{n+2}} \sum_{\substack{0 \leq x_n \leq q_n - 2, \\ 0 \leq x_{n+1} < q_{n+1}}} \varphi \left(h_n \left(1 + \left| \sigma_{x_{n+1}}^{(n)}(1 + x_n) - \sigma_{x_{n+1}}^{(n)}(x_n) \right| \right) \right) < \infty,$$

so S and T are φ -integrably orbit equivalent. \square

Remark 6.4. As Theorem C, the odomutants T in Theorem A and C can be built as homeomorphisms, with a strong orbit equivalence between them and the odometers S . This is clear for Theorem A since we may assume $\sigma^{(n,i)}(q_n - 1) = q_n - 1$ without loss of generality. For Theorem D, we have to slightly modify the settings in Lemma 6.2 and its proof. For example, we can define $\sigma_{x_{n+1}}^{(n)}$ as the permutation mapping 0 to 0, $q_n - 1$ to $q_n - 1$ and $i \in \{1, \dots, q_n - 2\}$ to $1 + (i - 1 + x_{n+1} \bmod q_n - 2)$. The set Y_1 becomes the set of $x \in X$ such that $x_n \notin \{0, q_n - 2, q_n - 1\}$ if n is even, $x_n \notin \{0, 1, q_n - 1\}$ if n is odd, and vice versa for Y_2 . Then the ideas remain the same.

A Some combinatorial properties

In this section, we fix an odomutant T built with uniformly \mathbf{c} -multiple permutations, with $\mathbf{c} = (c_n, \tilde{q}_n)_{n \geq 0}$ and $q_n := c_n \tilde{q}_n$. We refer the reader to Definition 3.3 for all the notations that we will use, although not defined in this section (for instance the partitions $\tilde{\mathcal{P}}(\ell)$, the segments $I_j^{(\ell)}$, etc).

In the proof of Theorem C, for combinatorial purposes appearing in the computation of topological entropy, we need to understand the dynamics of this odomutant with respect to the associated partition $\tilde{\mathcal{P}}(\ell)$ for some ℓ . Indeed, as explained in Example 2.3, computing the topological entropy with respect to a clopen partition partly consists in counting words given by the associated coding map. Recall that, given $c_n = 1$ for every $n \geq 0$, and an odomutant built with \mathbf{c} -multiple permutations, $\tilde{\mathcal{P}}(\ell)$ is the partition $\mathcal{P}(\ell)$ in ℓ -cylinders of the space $X = \prod_{n \geq 0} \{0, \dots, q_n - 1\}$, as introduced in Example 2.3.

As we can notice in the proofs of the following results, it is more convenient for the computations that the permutations have common fixed points (here this is the point 0), as in Section 3.5 when one wants to extend an odomutant to a homeomorphism. With this assumption, at each step of the cutting-and-stacking construction, we can study the words

produced by the points in the first level of the towers, and the recurrence relation describing such a word at step $n + 1$ as a concatenation of words at step n (Lemmas A.1 and A.3). Counting only these words gives a lower bound of the number of all the words produced by the coding map, thus providing a lower bound of the topological entropy with respect to the clopen partition that we consider. If this lower bound of $h_{\text{top}}(T)$ diverges to $+\infty$, then we have built an odomutant of infinite entropy. This is the strategy that we will apply in the proof of Theorem C in the case $\alpha = +\infty$, using a lower bound on the number of words provided by Lemma A.4 when the odomutant satisfies some assumptions. Note that this lemma is a reformulation of the main ideas of Boyle and Handelmann for the proof of their similar statement [BH94, Section 3]. In the case $\alpha < +\infty$, we will need an exact formula on the entropy. To this purpose, Lemma A.6 provides an upper bound of the number of all words produced by a coding map, and thus a finer upper bound of the entropy as we see in the proof of Theorem C.

Lemma A.1. *Let $\ell \geq 1$ and T be an odomutant built with uniformly c -multiple permutations fixing 0.*

1. *For every $n \geq \ell - 1$, for every $x_n \in \{0, 1, \dots, q_n - 1\}$, the set*

$$\{[\tilde{\mathcal{P}}(\ell)]_{h_n}(x) \mid x \in [0, \dots, 0, x_n]_{n+1}\}$$

is a singleton, denoted by $\{W(\tilde{\mathcal{P}}(\ell))_{x_n}^{(n)}\}$.

2. *The following holds in the case $n = \ell - 1$: the preimages of the map*

$$x_{\ell-1} \in \{0, 1, \dots, q_{\ell-1} - 1\} \mapsto W(\tilde{\mathcal{P}}(\ell))_{x_{\ell-1}}^{(\ell-1)}$$

are $I_0^{(\ell-1)}, \dots, I_{\tilde{q}_{\ell-1}}^{(\ell-1)}$. Therefore this map is $c_{\ell-1}$ -to-1.

3. *For every $n < \ell - 1$, for every $(x_n, \dots, x_{\ell-1}) \in \prod_{n \leq i \leq \ell-1} \{0, 1, \dots, q_i - 1\}$, the set*

$$\{[\tilde{\mathcal{P}}(\ell)]_{h_n}(x) \mid x \in [0, \dots, 0, x_n, \dots, x_{\ell-1}]_{\ell}\}$$

is a singleton, denoted by $\{W(\tilde{\mathcal{P}}(\ell))_{x_n, \dots, x_{\ell-1}}^{(n)}\}$.

4. *For every $n < \ell - 1$ and every $(x_n, \dots, x_{\ell-2}) \in \prod_{n \leq i \leq \ell-2} \{0, 1, \dots, q_i - 1\}$, the preimages of the map*

$$x_{\ell-1} \in \{0, 1, \dots, q_{\ell-1} - 1\} \mapsto W(\tilde{\mathcal{P}}(\ell))_{x_n, \dots, x_{\ell-2}, x_{\ell-1}}^{(n)}$$

are $I_0^{(\ell-1)}, \dots, I_{\tilde{q}_{\ell-1}}^{(\ell-1)}$. Therefore this map is $c_{\ell-1}$ -to-1.

Remark A.2.

- In the case of multiple permutations with $c_n = 1$ for every $n \geq 0$ (so $\tilde{q}_n = q_n$), we get $\tilde{\mathcal{P}}(\ell) = \mathcal{P}(\ell)$ and $I_j^{(\ell)} = \{j\}$ for every $\ell \geq 1$ and every $j \in \{0, \dots, q_{\ell} - 1\}$, so the map

$$x_{\ell-1} \in \{0, 1, \dots, q_{\ell-1} - 1\} \mapsto W(\mathcal{P}(\ell))_{x_{\ell-1}}^{(\ell-1)}$$

is injective.

- The first point of the above lemma remains true if we replace $\tilde{\mathcal{P}}(\ell)$ by any partition \mathcal{P} refined by $\mathcal{P}(\ell)$. Indeed, the result is true for the partition $\mathcal{P}(\ell)$ (it suffices to consider T as an odomutant built with multiple permutations and $c_n = 1$). Moreover, the word $[\tilde{\mathcal{P}}]_{h_n}(x)$ is obtained from the word $[\tilde{\mathcal{P}}(\ell)]_{h_n}(x)$ by applying letters by letters the projection which maps $P \in \tilde{\mathcal{P}}(\ell)$ to the atom of \mathcal{P} containing P .

Proof of Lemma A.1. Let $x \in [0, \dots, 0, x_n]_{n+1}$. We can write $x = (\underbrace{0, \dots, 0}_{n \text{ times}}, x_n, x_{n+1}, \dots)$. All

the permutations fix 0, so for every $i \geq n-1$, we have

$$\psi_i(x) = (\underbrace{0, \dots, 0}_{n \text{ times}}, \sigma_{x_{n+1}}^{(n)}(x_n), \dots, \sigma_{x_{i+1}}^{(i)}(x_i), x_{i+1}, x_{i+2}, \dots).$$

For $k \in \{0, 1, \dots, h_n - 1\}$, let $(k_0, k_1, \dots, k_{n-1})$ be the n -tuple in $\prod_{0 \leq i \leq n-1} \{0, 1, \dots, q_i - 1\}$ satisfying

$$k = k_0 + h_1 k_1 + \dots + h_{n-1} k_{n-1}.$$

We then have

$$S^k \psi_i(x) = (k_0, k_1, \dots, k_{n-1}, \sigma_{x_{n+1}}^{(n)}(x_n), \dots, \sigma_{x_{i+1}}^{(i)}(x_i), x_{i+1}, x_{i+2}, \dots)$$

so $T^k x$ is equal to $(y_0^{(k)}, \dots, y_{n-1}^{(k)}, x_n, x_{n+1}, \dots)$ where $y_i^{(k)}$ defined by

$$y_n^{(k)} = x_n, \\ \forall 0 \leq i \leq n-1, y_i^{(k)} = \left(\sigma_{y_{i+1}^{(k)}}^{(i)} \right)^{-1} (k_i).$$

Denote by $j(k, \ell-1)$ the integer in $\{0, 1, \dots, \tilde{q}_{\ell-1}\}$ satisfying $y_{\ell-1}^{(k)} \in I_{j(k, \ell-1)}^{(\ell-1)}$. For every $k \in \{0, 1, \dots, h_n - 1\}$, $(y_0^{(k)}, \dots, y_n^{(k)})$ does not depend on x_{n+1}, x_{n+2}, \dots and only depends on x_n , so does the h_n -tuple $([y_0^{(k)}, \dots, y_{\ell-2}^{(k)}, I_{j(k, \ell-1)}^{(\ell-1)}]_{0 \leq k \leq h_n-1})$ which is equal to $[\tilde{\mathcal{P}}(\ell)]_{h_n}(x)$.

In the case $n = \ell-1$, we have $y_{\ell-1}^{(k)} = x_n$, so the value of the word $W(\tilde{\mathcal{P}}(\ell))_{x_n}^{(n)}$ only depends on the interval $I_j^{(n)}$ containing x_n .

We similarly prove the last two items. \square

Lemma A.3. *Let $\ell \geq 1$ and T be an odomutant built with uniformly \mathbf{c} -multiple permutations fixing 0. Let us recall the words $W(\tilde{\mathcal{P}}(\ell))_{x_n}^{(n)}$ defined in Lemma A.1. Then, for every $n \geq \ell-1$ and $x_n \in \{0, 1, \dots, q_n - 1\}$, we have*

$$W(\tilde{\mathcal{P}}(\ell))_{x_{n+1}}^{(n+1)} = W(\tilde{\mathcal{P}}(\ell))_0^{(n)} \cdot W(\tilde{\mathcal{P}}(\ell))_{\left(\sigma_{x_{n+1}}^{(n)}\right)^{-1}(1)}^{(n)} \cdot \dots \cdot W(\tilde{\mathcal{P}}(\ell))_{\left(\sigma_{x_{n+1}}^{(n)}\right)^{-1}(q_n-1)}^{(n)}.$$

Proof of Lemma A.3. Given $n \geq \ell-1$, note that we have

$$\{0, 1, \dots, h_{n+1} - 1\} = \bigsqcup_{0 \leq i < q_n} \left(\{0, 1, \dots, h_n - 1\} + h_n i \right).$$

Moreover if i is in $\{0, 1, \dots, q_n - 1\}$, if x_{n+1} is in $\{0, 1, \dots, q_{n+1} - 1\}$, we have

$$T^{ih_n}([0, \dots, 0, 0, x_{n+1}]_{n+2}) = [0, \dots, 0, \left(\sigma_{x_{n+1}}^{(n)}\right)^{-1}(i), x_{n+1}]_{n+2}.$$

This implies that, for a fixed $x \in [0, \dots, 0, 0, x_{n+1}]_{n+2}$, the element $y_i := T^{ih_n}(x)$ is in $[0, \dots, 0, (\sigma_{x_{n+1}}^{(n)})^{-1}(i)]_{n+1}$ and we get

$$[\tilde{\mathcal{P}}(\ell)]_{ih_n, (i+1)h_n-1}(x) = [\tilde{\mathcal{P}}(\ell)]_{h_n}(T^{ih_n}(x)) = [\tilde{\mathcal{P}}(\ell)]_{h_n}(y_i) = W(\tilde{\mathcal{P}}(\ell))_{(\sigma_{x_{n+1}}^{(n)})^{-1}(i)}^{(n)}$$

by Lemma A.1. Finally the h_{n+1} -word on x is the following concatenation:

$$\begin{aligned} W(\tilde{\mathcal{P}}(\ell))_{x_{n+1}}^{(n)} &= [\tilde{\mathcal{P}}(\ell)]_{h_{n+1}}(x) \\ &= [\tilde{\mathcal{P}}(\ell)]_{0, h_{n+1}-1}(x) \\ &= [\tilde{\mathcal{P}}(\ell)]_{0, h_n-1}(x) \cdot [\tilde{\mathcal{P}}(\ell)]_{h_n, 2h_n-1}(x) \cdot \dots \cdot [\tilde{\mathcal{P}}(\ell)]_{h_n(q_{n+1}-1), h_{n+1}-1}(x) \\ &= W(\tilde{\mathcal{P}}(\ell))_{(\sigma_{x_{n+1}}^{(n)})^{-1}(0)}^{(n)} \cdot W(\tilde{\mathcal{P}}(\ell))_{(\sigma_{x_{n+1}}^{(n)})^{-1}(1)}^{(n)} \cdot \dots \cdot W(\tilde{\mathcal{P}}(\ell))_{(\sigma_{x_{n+1}}^{(n)})^{-1}(q_n-1)}^{(n)} \\ &= W(\tilde{\mathcal{P}}(\ell))_0^{(n)} \cdot W(\tilde{\mathcal{P}}(\ell))_{(\sigma_{x_{n+1}}^{(n)})^{-1}(1)}^{(n)} \cdot \dots \cdot W(\tilde{\mathcal{P}}(\ell))_{(\sigma_{x_{n+1}}^{(n)})^{-1}(q_n-1)}^{(n)} \end{aligned}$$

and we are done. \square

Lemma A.4. *Let T be an odomutant built with uniformly \mathbf{c} -multiple permutations $\tau_j^{(n)}$ fixing 0. Let $(\kappa_n)_{n \geq 1}$ be a sequence of positive integers and assume that for every $n \geq 0$, the map*

$$j \in \{0, 1, \dots, \tilde{q}_{n+1} - 1\} \mapsto (\tau_j^{(n)}(I_0^{(n)}), \dots, \tau_j^{(n)}(I_{\tilde{q}_n-1}^{(n)}))$$

is κ_{n+1} -to-1 (in particular, κ_{n+1} divides \tilde{q}_{n+1})¹⁸. Then, for all $n \geq \ell \geq 1$, we have

$$\left| \left\{ W(\tilde{\mathcal{P}}(\ell))_{x_n}^{(n)} \mid 0 \leq x_n \leq q_n - 1 \right\} \right| \geq \frac{\tilde{q}_n}{\prod_{k=\ell}^n \kappa_k^{h_n/h_k}}.$$

Remark A.5. In the case of uniform permutations with pairwise different permutations, the lemma implies that

$$\left| \left\{ W(\tilde{\mathcal{P}}(\ell))_{x_n}^{(n)} \mid 0 \leq x_n \leq q_n - 1 \right\} \right| = q_n$$

so $W(\tilde{\mathcal{P}}(\ell))_{x_n}^{(n)}$ is an injective function of x_n . This could also be deduced from Lemma A.3. Therefore, odomutants can have more words in their language than odometer, and then their entropy can be positive.

Proof of Lemma A.4. Let $(\tilde{\mathcal{P}}(\ell))_{\ell \geq 1}$ be the sequence of partitions associated to the construction of this odomutant with uniformly \mathbf{c} -multiple permutations. Given $n \geq \ell \geq 1$, we consider the projection $\pi_{n+1, \ell}: \tilde{\mathcal{P}}(n+1) \rightarrow \tilde{\mathcal{P}}(\ell)$ which maps $P \in \tilde{\mathcal{P}}(n+1)$ to the atom of $\tilde{\mathcal{P}}(\ell)$ containing P . This projection induces a map on the set of words with letters in $\tilde{\mathcal{P}}(n+1)$, it consists in projecting each entry on $\tilde{\mathcal{P}}(\ell)$.

¹⁸Let us go back to the intuition behind uniformly multiple permutations. Since we consider the partitions $\tilde{\mathcal{P}}(\ell)$ instead of $\mathcal{P}(\ell)$, we cannot distinguish between the copies of a subcolumn that we stack to form each tower. Therefore, given two permutations $\tau_j^{(n)}$ and $\tau_{j'}^{(n)}$, if $(\tau_j^{(n)}(I_0^{(n)}), \dots, \tau_j^{(n)}(I_{\tilde{q}_n-1}^{(n)})) = (\tau_{j'}^{(n)}(I_0^{(n)}), \dots, \tau_{j'}^{(n)}(I_{\tilde{q}_n-1}^{(n)}))$, then we cannot distinguish between the permutations that they encode, although these permutations are different.

Claim 1. Let $x \in [0, \dots, 0]_n$ and $k \in \{1, \dots, n\}$. For every $i \in \{0, 1, \dots, \frac{h_n}{h_k} - 1\}$ and every $j \in \{0, 1, \dots, q_{k-1} - 1\}$, the point $x^{(i,j)} := T^{ih_k + jh_{k-1}}x$ is in $[0, \dots, 0]_{k-1}$. Moreover, $(x^{(i,j)})_k$ does not depend on j and we have

$$(x^{(i,j)})_k = \sigma_{(x^{(i,j)})_{k+1}}^{(k)}(i_k) \text{ and } (x^{(i,j)})_{k-1} = \sigma_{(x^{(i,j)})_k}^{(k-1)}(j)$$

where $i_k := \left\lfloor \frac{i}{q_k} \right\rfloor$.

Proof of the claim. Let us write $ih_k = i_k h_k + i_{k+1} h_{k+1} + \dots + i_{n-1} h_{n-1}$ with $i_m \in \{0, \dots, q_m - 1\}$ for every $m \in \{k, \dots, n-1\}$. Given $j \geq n$, we have

$$\psi_j(x) = (\underbrace{0, \dots, 0}_{n \text{ times}}, \sigma_{x_{n+1}}^{(n)}(x_n), \dots)$$

and

$$S^{ih_k + jh_{k-1}}\psi_j(x) = (\underbrace{0, \dots, 0}_{k-1 \text{ times}}, j, i_k, \dots, i_{n-1}, \sigma_{x_{n+1}}^{(n)}(x_n), \dots).$$

Hence we get $x^{(i,j)} = \psi_j^{-1} S^{ih_k + jh_{k-1}} \psi_j(x)$ for every $j \geq n$, which implies

$$\begin{aligned} (x^{(i,j)})_n &= x_n, \\ (x^{(i,j)})_{n-1} &= \sigma_{x_n}^{(n-1)}(i_{n-1}), \\ (x^{(i,j)})_{n-2} &= \sigma_{(x^{(i,j)})_{n-1}}^{(n-2)}(i_{n-2}), \\ &\vdots \\ (x^{(i,j)})_k &= \sigma_{(x^{(i,j)})_{k+1}}^{(k)}(i_k), \\ (x^{(i,j)})_{k-1} &= \sigma_{(x^{(i,j)})_k}^{(k-1)}(j), \end{aligned}$$

so we are done. □claim

Claim 2. With the hypotheses of the lemma, for every $k \in \{\ell, \dots, n\}$, the map

$$\pi_{k+1,k}: \left\{ W(\tilde{\mathcal{P}}(k+1))_{x_n}^{(n)} \mid 0 \leq x_n \leq q_n - 1 \right\} \rightarrow \left\{ W(\tilde{\mathcal{P}}(k))_{x_n}^{(n)} \mid 0 \leq x_n \leq q_n - 1 \right\}$$

is at most $\kappa_k^{h_n/h_k}$ -to-1.

Proof of the claim. Let $x \in [0, \dots, 0]_n$. We have $[\tilde{\mathcal{P}}(k+1)]_{h_n}(x) = W(\tilde{\mathcal{P}}(k+1))_{x_n}^{(n)}$ and $[\tilde{\mathcal{P}}(k)]_{h_n}(x) = W(\tilde{\mathcal{P}}(k))_{x_n}^{(n)}$. We first write these words as a concatenation of words of size h_{k-1} , namely the words $[\tilde{\mathcal{P}}(k+1)]_{h_{k-1}}(T^{mh_{k-1}}x)$ or $[\tilde{\mathcal{P}}(k)]_{h_{k-1}}(T^{mh_{k-1}}x)$ for $m \in \{0, 1, \dots, \frac{h_n}{h_{k-1}} - 1\}$. Given $m \in \{0, 1, 2, \dots, \frac{h_n}{h_{k-1}} - 1\}$, the point $T^{mh_{k-1}}(x)$ is in $[0, \dots, 0]_{k-1}$ by the last claim, so we have

$$[\tilde{\mathcal{P}}(k+1)]_{h_{k-1}}(T^{mh_{k-1}}(x)) = W(\tilde{\mathcal{P}}(k+1))_{(T^{mh_{k-1}}(x))_{k-1}, (T^{mh_{k-1}}(x))_k}^{(k-1)}$$

and

$$[\tilde{\mathcal{P}}(k)]_{h_{k-1}}(T^{mh_{k-1}}(x)) = W(\tilde{\mathcal{P}}(k))_{(T^{mh_{k-1}}(x))_{k-1}}^{(k-1)}.$$

Secondly, we gather these words of length h_{k-1} in groups $M(k+1)_{x_n,i}$ or $M(k)_{x_n,i}$ of q_{k-1} words:

$$M(k+1)_{x_n,i} := W(\tilde{\mathcal{P}}(k+1))_{(x^{(i,0)})_{k-1}, (x^{(i,0)})_k}^{(k-1)} \cdot W(\tilde{\mathcal{P}}(k+1))_{(x^{(i,1)})_{k-1}, (x^{(i,1)})_k}^{(k-1)} \\ \cdot \dots \cdot W(\tilde{\mathcal{P}}(k+1))_{(x^{(i,q_{k-1}-1)})_{k-1}, (x^{(i,q_{k-1}-1)})_k}^{(k-1)}$$

and

$$M(k)_{x_n,i} := W(\tilde{\mathcal{P}}(k))_{(x^{(i,0)})_{k-1}}^{(k-1)} \cdot W(\tilde{\mathcal{P}}(k))_{(x^{(i,1)})_{k-1}}^{(k-1)} \cdot \dots \cdot W(\tilde{\mathcal{P}}(k))_{(x^{(i,q_{k-1}-1)})_{k-1}}^{(k-1)}$$

for all $i \in \{0, 1, \dots, \frac{h_n}{h_k} - 1\}$ in such a way that we have

$$W(\tilde{\mathcal{P}}(k+1))_{x_n}^{(n)} = M(k+1)_{x_n,0} \cdot M(k+1)_{x_n,1} \cdot \dots \cdot M(k+1)_{x_n, \frac{h_n}{h_k}-1}$$

and

$$W(\tilde{\mathcal{P}}(k))_{x_n}^{(n)} = M(k)_{x_n,0} \cdot M(k)_{x_n,1} \cdot \dots \cdot M(k)_{x_n, \frac{h_n}{h_k}-1}.$$

To prove the lemma, it now remains to prove that, for every $i \in \{0, 1, \dots, \frac{h_n}{h_k} - 1\}$, the map

$$\pi_{k+1,k}: \{M(k+1)_{x_n,i} \mid 0 \leq x_n \leq q_n - 1\} \rightarrow \{M(k)_{x_n,i} \mid 0 \leq x_n \leq q_n - 1\}$$

is at most κ_k -to-1. Let us fix a word $M(k)_{x_n,i}$ with $i \in \{0, 1, \dots, \frac{h_n}{h_k} - 1\}$ and $x_n \in \{0, 1, \dots, q_n - 1\}$. We write $i_k = \lfloor i/q_k \rfloor$. By the last claim, the quantities $(x^{(i,0)})_k, \dots, (x^{(i,q_{k-1}-1)})_k$ are equal and their common value is denoted by X_k , and we have

$$(x^{(iq_{k-1},j)})_{k-1} = \left(\sigma_{X_k}^{(k-1)} \right)^{-1}(j) \quad (7)$$

for every $j \in \{0, 1, \dots, q_{k-1}\}$. This first implies that

$$(x^{(i,0)})_{k-1}, (x^{(i,1)})_{k-1}, \dots, (x^{(i,q_{k-1}-1)})_{k-1}$$

are q_{k-1} pairwise different elements of $\{0, 1, \dots, q_{k-1} - 1\}$. Since we know each subword $W(\tilde{\mathcal{P}}(k))_{(x^{(i,j)})_{k-1}}^{(k-1)}$ of $M(k)_{x_n,i}$, the third item of Lemma A.1 implies that we completely know the sets $I_0^{(k-1)}, \dots, I_{q_{k-1}-1}^{(k-1)}$, so

$$\left(\sigma_{X_k}^{(k-1)}(I_0^{(k-1)}), \dots, \sigma_{X_k}^{(k-1)}(I_{q_{k-1}-1}^{(k-1)}) \right)$$

is also completely determined. By assumptions, X_k is in the disjoint union of κ_k sets of the form $I_j^{(k)}$.

To conclude, we have proved that, if we have $\pi_{k+1,k}(M(k+1)_{y_n,i}) = M(k)_{x_n,i}$ for some $y_n \in \{0, 1, \dots, q_n - 1\}$, then $M(k+1)_{y_n,i}$ is of the form

$$W(\tilde{\mathcal{P}}(k+1))_{(x^{(i,0)})_{k-1}, X_k}^{(k-1)} \cdot W(\tilde{\mathcal{P}}(k+1))_{(x^{(i,1)})_{k-1}, X_k}^{(k-1)} \cdot \dots \cdot W(\tilde{\mathcal{P}}(k+1))_{(x^{(i,q_{k-1}-1)})_{k-1}, X_k}^{(k-1)}$$

with X_k in the disjoint union of κ_k sets of the form $I_j^{(k)}$, and which completely determines $(x^{(i,0)})_{k-1}, (x^{(i,1)})_{k-1}, \dots, (x^{(i,q_{k-1}-1)})_{k-1}$ by Equality (7). But since two elements X_k and X'_k in the same $I_j^{(k)}$ provide the same word $M(k+1)_{y_n,i}$ (by the last item of Lemma A.1), we get that there are at most κ_k possible values for the word $M(k+1)_{y_n,i}$. \square claim

By the last claim, the map

$$\pi_{n+1,\ell}: \left\{ W(\tilde{\mathcal{P}}(n+1))_{x_n}^{(n)} \mid 0 \leq x_n \leq q_n - 1 \right\} \rightarrow \left\{ W(\tilde{\mathcal{P}}(\ell))_{x_n}^{(n)} \mid 0 \leq x_n \leq q_n - 1 \right\}$$

is at most $\left(\prod_{k=\ell}^n \kappa_k^{h_n/h_k} \right)$ -to-1, so we have

$$\left| \left\{ W(\tilde{\mathcal{P}}(\ell))_{x_n}^{(n)} \mid 0 \leq x_n \leq q_n - 1 \right\} \right| \geq \frac{\left| \left\{ W(\tilde{\mathcal{P}}(n+1))_{x_n}^{(n)} \mid 0 \leq x_n \leq q_n - 1 \right\} \right|}{\prod_{k=\ell}^n \kappa_k^{h_n/h_k}}$$

and the result follows from the second item of Lemma A.1. \square

Lemma A.6. *Let $(q_n)_{n \geq 0}$ be a sequence of integers greater than or equal to 2 and satisfying $q_{n+1} \leq (q_n - 2)!$. Then there exist permutations $\sigma_{x_{n+1}}^{(n)}$, for $n \geq 0$ and $x_{n+1} \in \{0, \dots, q_{n+1} - 1\}$, satisfying the following properties:*

1. *for every $n \geq 0$, the maps $\sigma_0^{(n)}, \sigma_1^{(n)}, \dots, \sigma_{q_{n+1}-1}^{(n)}$ are pairwise different permutations of the set $\{0, \dots, q_n - 1\}$, fixing 0 and $q_n - 1$;*
2. *the following bounds hold for every $n \geq \ell \geq 1$:*

$$q_n \leq N(\mathcal{P}(\ell)_0^{h_n-1}) \leq h_{n-1} q_n q_{n-1}^2 2^{q_{n-1}}$$

Proof of Lemma A.6. Let us recall that $N(\mathcal{P}(\ell)_0^{h_n-1})$ is equal to the cardinality of $\{[\mathcal{P}(\ell)]_{h_n}(x) \mid x \in X\}$. If the permutations $\sigma_0^{(n)}, \sigma_1^{(n)}, \dots, \sigma_{q_{n+1}-1}^{(n)}$ are pairwise different for every $n \geq 0$, then we get $N(\mathcal{P}(\ell)_0^{h_n-1}) \geq q_n$ for the underlying odomutant (see Remark A.5 following Lemma A.4).

Given $n \geq 0$, let $i_n \in \{2, \dots, q_n - 2\}$ be such that $(i_n - 1)! < q_{n+1} \leq i_n!$ and let us choose any family $\left(\sigma_{x_{n+1}}^{(n)} \right)_{0 \leq x_{n+1} < q_{n+1}}$ of pairwise different permutations of the set $\{0, \dots, q_n - 1\}$ fixing $0, i_n + 1, i_n + 2, \dots, q_n - 1$. Given an integer $\ell \geq 1$, let us find an upper bound of $N(\mathcal{P}(\ell)_0^{h_n-1})$ for every $n \geq \ell$. Let $n \geq \ell$ and $x \in X$. There exists $i \in \{0, 1, \dots, h_n - 1\}$ such that $y := T^{-i}x$ is in $[0, \dots, 0, x_n]_{n+1}$. Let us write $z := T^{h_n}x$. Thus $[\mathcal{P}(\ell)]_{h_n}(x)$ is the concatenation of a final subword of $W(\mathcal{P}(\ell))_{x_n}^{(n)}$ and an initial subword of $W(\mathcal{P}(\ell))_{z_n}^{(n)}$. Writing $i = jh_{n-1} + r$ with integers $j \in \{0, \dots, q_{n-1} - 1\}$ and $r \in \{0, \dots, h_{n-1} - 1\}$, and using Lemma A.3, we have

$$\begin{aligned} [\mathcal{P}(\ell)]_{h_n}(x) &= w \cdot W(\mathcal{P}(\ell))_{\left(\sigma_{x_n}^{(n-1)}\right)^{-1}(j+1)}^{(n-1)} \cdot \dots \cdot W(\mathcal{P}(\ell))_{\left(\sigma_{x_n}^{(n-1)}\right)^{-1}(q_{n-1}-2)}^{(n-1)} \cdot W(\mathcal{P}(\ell))_{q_{n-1}-1}^{(n-1)} \\ &\quad \cdot W(\mathcal{P}(\ell))_0^{(n-1)} \cdot W(\mathcal{P}(\ell))_{\left(\sigma_{z_n}^{(n-1)}\right)^{-1}(1)}^{(n-1)} \cdot \dots \cdot W(\mathcal{P}(\ell))_{\left(\sigma_{z_n}^{(n-1)}\right)^{-1}(j-1)}^{(n-1)} \cdot w' \end{aligned}$$

where w is a final subword of $W(\mathcal{P}(\ell))_{\left(\sigma_{x_n}^{(n-1)}\right)^{-1}(j)}^{(n-1)}$ of length $h_{n-1} - r$, and w' an initial subword of $W(\mathcal{P}(\ell))_{\left(\sigma_{z_n}^{(n-1)}\right)^{-1}(j)}^{(n-1)}$ of length r . Therefore, to every word of the form $[\mathcal{P}(\ell)]_{h_n}(x)$ for the points $x \in X$ sharing the same integers x_n, z_n, j and r , we can associate the family

$$\left(\left(\sigma_{x_n}^{(n-1)} \right)^{-1}(j), \dots, \left(\sigma_{x_n}^{(n-1)} \right)^{-1}(q_{n-1} - 2), \left(\sigma_{z_n}^{(n-1)} \right)^{-1}(1), \dots, \left(\sigma_{z_n}^{(n-1)} \right)^{-1}(j) \right).$$

In the particular cases $j = 0$ and $j = q_{n-1} - 1$, this family is respectively equal to

$$\left(\left(\sigma_{x_n}^{(n-1)} \right)^{-1} (1), \dots, \left(\sigma_{x_n}^{(n-1)} \right)^{-1} (q_{n-1} - 2) \right)$$

and

$$\left(\left(\sigma_{z_n}^{(n-1)} \right)^{-1} (1), \dots, \left(\sigma_{z_n}^{(n-1)} \right)^{-1} (q_{n-1} - 2) \right).$$

Moreover, this association is injective, as a consequence of Remark A.2 following Lemma A.1. Thus we have

$$\begin{aligned} N(\mathcal{P}(\ell)_0^{h_{n-1}}) &\leq \sum_{r=0}^{h_{n-1}-1} \left(a_1^{(n)} + b_{q_{n-1}-2}^{(n)} + \sum_{j=1}^{q_{n-1}-2} a_j^{(n)} \times b_j^{(n)} \right) \\ &\leq h_{n-1} \left(a_1^{(n)} + b_{q_{n-1}-2}^{(n)} + \sum_{j=1}^{q_{n-1}-2} a_j^{(n)} \times b_j^{(n)} \right) \end{aligned}$$

where $a_j^{(n)}$ and $b_j^{(n)}$ are respectively the cardinality of

$$\left\{ \left(\left(\sigma_{x_n}^{(n-1)} \right)^{-1} (j), \dots, \left(\sigma_{x_n}^{(n-1)} \right)^{-1} (q_{n-1} - 2) \right) \mid x_n \in \{0, 1, \dots, q_n - 1\} \right\}$$

and

$$\left\{ \left(\left(\sigma_{x_n}^{(n-1)} \right)^{-1} (1), \dots, \left(\sigma_{x_n}^{(n-1)} \right)^{-1} (j) \right) \mid x_n \in \{0, 1, \dots, q_n - 1\} \right\}.$$

We now find upper bounds of the quantities $a_j^{(n)}$ and $b_j^{(n)}$, using the properties of the permutations that we have chosen at the beginning of this proof. We have $a_1^{(n)} = q_n$, $a_j^{(n)} \leq i_{n-1} \times \dots \times j$ if $1 \leq j \leq i_{n-1}$ and $a_j^{(n)} = 1$ if $i_{n-1} + 1 \leq j \leq q_{n-1} - 2$. We also have $b_j^{(n)} \leq i_{n-1} \times \dots \times (i_{n-1} - j + 1)$ if $1 \leq j \leq i_{n-1} - 1$ and $b_j^{(n)} = q_n$ if $i_{n-1} \leq j \leq q_{n-1} - 2$. We then get

$$\begin{aligned} &a_1^{(n)} + b_{q_{n-1}-2}^{(n)} + \sum_{j=1}^{q_{n-1}-2} a_j^{(n)} \times b_j^{(n)} \\ &\leq 2q_n + \left(\sum_{j=1}^{i_{n-1}-1} \frac{i_{n-1}!}{(j-1)!} \frac{i_{n-1}!}{(i_{n-1}-j)!} \right) + q_n i_{n-1} + \left(\sum_{j=i_{n-1}+1}^{q_{n-1}-2} q_n \right) \\ &\leq q_n q_{n-1} + i_{n-1}^2 (i_{n-1} - 1)! \sum_{j=1}^{i_{n-1}-1} \binom{i_{n-1}-1}{j-1} \\ &\leq q_n q_{n-1} + q_{n-1}^2 q_n 2^{i_{n-1}-1} \\ &\leq q_n q_{n-1}^2 2^{i_{n-1}} \\ &\leq q_n q_{n-1}^2 2^{q_{n-1}}, \end{aligned}$$

and we are done □

B Further comments on odomutants: Bratteli diagrams, strong orbit equivalence

B.1 Bratteli diagrams, strong orbit equivalence

We introduce the most important definitions and results in the context of strong orbit equivalence. For more details, we refer the reader to [HPS92] and [GPS95].

B.1.1 Bratteli diagrams

A **Bratteli diagram** is a graph $B = (V, E)$ with the set of vertices

$$V = \bigsqcup_{k \geq 0} V_k$$

and the set of edges

$$E = \bigsqcup_{k \geq 0} E_k,$$

where V_k and E_k are finite, $V_0 = \{v^{(0)}\}$ and the edges in E_k connect vertices in V_k to vertices in V_{k+1} (multiple edges between two vertices are allowed). If $e_k \in E_k$ connects $v_k \in V_k$ to $v_{k+1} \in V_{k+1}$, we write $s(e_k) = v_k$ and $r(e_k) = v_{k+1}$, this provides maps $s: E \rightarrow V$ (**source map**) satisfying $s(E_k) \subset V_k$ and $r: E \rightarrow V$ (**range map**) satisfying $r(E_k) \subset V_{k+1}$. We assume that

$$\forall v \in V, s^{-1}(v) \neq \emptyset$$

and

$$\forall v \in V \setminus V_0, r^{-1}(v) \neq \emptyset.$$

For $k < \ell$, a **path** from $v_k \in V_k$ to $v_\ell \in V_\ell$ is a tuple $(e_k, e_{k+1}, \dots, e_{\ell-1})$ satisfying $s(e_k) = v_k$, $r(e_i) = s(e_{i+1})$ for every $i \in \{k, \dots, \ell-2\}$ and $r(e_{\ell-1}) = v_\ell$.

An **ordered** Bratteli diagram is a Bratteli diagram together with a linear order in $r^{-1}(v)$ for every $v \in V \setminus V_0$, namely we consider a bijection

$$r^{-1}(v) \rightarrow \{0, 1, \dots, |r^{-1}(v)| - 1\}$$

for every $v \in V \setminus V_0$. Then we consider E_k as a subset of $V_k \times V_{k+1} \times \mathbb{N}$: an edge $e_k \in E_k$ is written as (v_k, v_{k+1}, ρ_k) where $v_k = s(e_k)$, $v_{k+1} = r(e_k)$ and $\rho_k \in \{0, \dots, |r^{-1}(v_{k+1})| - 1\}$ is the **rank** of e_k for the linear order in $r^{-1}(v_{k+1})$, we write $\rho_k = \text{rk}(e_k)$.

Let us set

$$X_B := \left\{ (e_k)_{k \geq 0} \in \prod_{k \geq 0} E_k \mid \forall k \geq 0, r(e_k) = s(e_{k+1}) \right\},$$

$$X_{B, \min} := \{(e_k)_{k \geq 0} \in X_B \mid \forall k \geq 0, \text{rk}(e_k) = 0\}$$

$$\text{and } X_{B, \max} := \{(e_k)_{k \geq 0} \in X \mid \forall k \geq 0, \text{rk}(e_k) = |r^{-1}(r(e_k))| - 1\}.$$

As a subset of $\prod_{k \geq 0} E_k$, X_B is endowed with the induced product topology. X_B is a compact and totally disconnected metric space. By definition, the cylinders¹⁹ of X_B are clopen sets and form a basis of the topology.

¹⁹defined as in Section 2.5

A Bratteli diagram is **simple** if there exists a subsequence (k_n) such that for every pair of vertices in $V_{k_n} \times V_{k_{n+1}}$, there exists a path between them. If an ordered Bratteli diagram is simple, then X_B has no isolated points, so it is a Cantor set.

Given a Bratteli diagram $B = (V, E)$, we can enumerate the vertices of each V_n :

$$V_n = \{v_0^{(n)}, \dots, v_{|V_n|-1}^{(n)}\};$$

and define the **incidence matrices**

$$M_n := \left(m_{i,j}^{(n)} \right)_{\substack{0 \leq i \leq |V_{n+1}|-1 \\ 0 \leq j \leq |V_n|-1}}$$

where $m_{i,j}^{(n)}$ is the number of edges of E_n connecting $v_j^{(n)}$ to $v_i^{(n+1)}$.

B.1.2 Bratteli-Vershik systems

Given an ordered Bratteli diagram B , we define a map $T_B: X_B \setminus X_{B,\max} \rightarrow X_B \setminus X_{B,\min}$ in the following way.

Let $x = (e_k)_{k \geq 0} \in X_B \setminus X_{B,\max}$ and

$$N := \min\{i \geq 0 \mid \text{rk}(e_i) < |r^{-1}(r(e_i))| - 1\}.$$

Let f_N be the edge in $r^{-1}(r(e_N))$ satisfying $\text{rk}(f_N) = \text{rk}(e_N) + 1$ and (f_0, \dots, f_{N-1}) the minimal path from $v^{(0)}$ to $s(f_N)$, namely this is the unique path satisfying $\text{rk}(f_i) = 0$ for every $i \in \{0, \dots, N-1\}$.²⁰ Then we define

$$T_B x := (f_1, \dots, f_N, e_{N+1}, e_{N+2}, \dots).$$

The map T_B is called the **Bratteli-Vershik system** associated to the ordered Bratteli diagram B .

An ordered and simple Bratteli diagram is **properly ordered** if $X_{B,\min}$ and $X_{B,\max}$ are singletons. Given a properly ordered Bratteli diagram, we extend T to the whole set X_B by setting

$$T_B(x_{\max}) := x_{\min}$$

where $X_{B,\max} := \{x_{\max}\}$ and $X_{B,\min} = \{x_{\min}\}$. In this case, we can check that T_B is a Cantor minimal homeomorphism.

For example, the Bratteli-Vershik system of the diagram in Figure 5 is topologically conjugate to the odometer on $X := \prod_{n \geq 0} \{0, 1, \dots, q_n - 1\}$, the following map

$$\Psi: (x_n)_{n \geq 0} \in X \mapsto \left(\left(v^{(0)}, v^{(1)}, x_0 \right), \left(v^{(1)}, v^{(2)}, x_1 \right), \left(v^{(2)}, v^{(3)}, x_2 \right), \dots \right) \in X_B$$

is a conjugation between them. As we explain in the next part, every Cantor minimal homeomorphism can be described by a Bratteli diagram.

²⁰We find this path in an inductive way: f_{N-1} is the unique edge satisfying $r(f_{N-1}) = s(f_N)$ and $\text{rk}(f_{N-1}) = 0$, f_{N-2} is the unique edge satisfying $r(f_{N-2}) = s(f_{N-1})$ and $\text{rk}(f_{N-2}) = 0$, and so on.

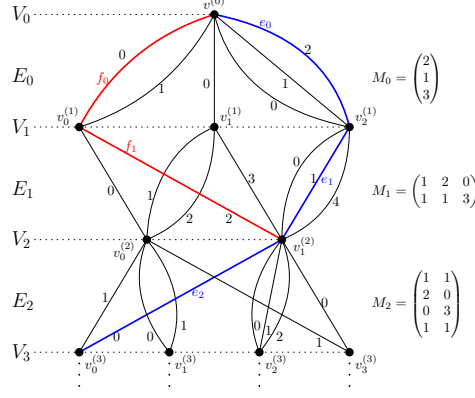


Figure 4: Example of ordered Bratteli diagram B . The image of (e_0, e_1, e_2, \dots) by T_B is (f_0, f_1, e_2, \dots) .

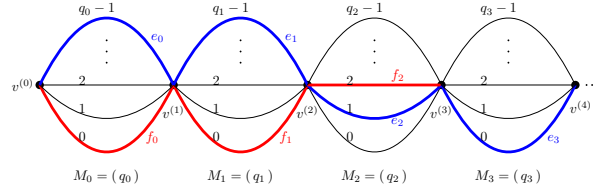


Figure 5: An ordered Bratteli diagram describing the odometer on $\prod_{n \geq 0} \{0, \dots, q_n - 1\}$. The image of $(e_0, e_1, e_2, e_3, \dots)$ by T_B is $(f_0, f_1, f_2, e_3, \dots)$.

B.1.3 Cantor minimal homeomorphisms

The Bratteli-Vershik systems of properly ordered Bratteli diagrams describe all the Cantor minimal homeomorphisms.

Theorem (Herman, Putnam, Skau [HPS92]). *If T is a Cantor minimal homeomorphism, then there exists a properly ordered Bratteli diagram B such that the associated Bratteli-Vershik system T_B is topologically conjugate to T .*

We briefly describe how a Cantor minimal homeomorphism $T: X \rightarrow X$ is encoded by a properly ordered Bratteli diagram. All we have to find is an increasing sequence $(\mathcal{P}_n)_{n \geq 0}$ of partitions generating the topology, of the form

$$\mathcal{P}_n := \{T^j(B_{n,i}) \mid 0 \leq i \leq k_n - 1, 0 \leq j \leq h_i^{(n)} - 1\}$$

where $k_n, h_1^{(n)}, \dots, h_{k_n}^{(n)}$ are positive integers, and the sequence $(B_n)_{n \geq 0}$ defined by

$$B_n := \bigsqcup_{0 \leq i \leq k_n - 1} B_{n,i},$$

is decreasing to a singleton $\{y\}$.

By "increasing sequence of partitions", we mean that \mathcal{P}_{n+1} is finer than \mathcal{P}_n , namely the atoms of \mathcal{P}_n are unions of atoms of \mathcal{P}_{n+1} . The partition \mathcal{P}_n is composed of k_n towers and given $i \in \{0, \dots, k_n - 1\}$, the tower

$$\mathcal{T}_{n,i} := \{B_{n,i}, T(B_{n,i}), \dots, T^{h_i^{(n)} - 1}(B_{n,i})\}$$

has height $h_i^{(n)}$.

Without the assumption that the partitions have to generate the topology, the construction only consists in considering in an inductive way a clopen subset B_{n+1} of B_n , that we partition in $B_{n,0}, \dots, B_{n,k_n-1}$ according to the value of the first return time. The underlying sequence of partitions does not necessarily generate the topology. For a generating sequence, we refer the reader to Lemma 4.1 of in [HPS92] and Lemma 3.1 in [Put89] for more details.

The properly ordered Bratteli diagram $B = (V, E)$ is defined as follows. Assume that $\mathcal{P}_0 = X$ (so $k_0 = h_1 = 1$) and define

$$V_n := \{\mathcal{T}_{n,i} \mid 0 \leq i \leq k_n - 1\}.$$

Given $n \geq 1$ and $i \in \{0, \dots, k_n - 1\}$, the tower $\mathcal{T}_{n,i}$ visits successively the towers

$$\mathcal{T}_{n-1,\ell_1^{(n,i)}}, \mathcal{T}_{n-1,\ell_2^{(n,i)}}, \dots, \mathcal{T}_{n-1,\ell_{r_{n,i}}^{(n,i)}}$$

with integers $\ell_j^{(n,i)} \in \{0, \dots, k_{n-1} - 1\}$ and $r_{n,i} \geq 1$. Then E is defined so that $r^{-1}(\mathcal{T}_{n,i})$ has cardinality $r_{n,i}$ and

$$r^{-1}(\mathcal{T}_{n,i}) := \left\{ \left(\mathcal{T}_{n-1,\ell_{j+1}^{(n,i)}}, \mathcal{T}_{n,i}, j \right) \mid 0 \leq j \leq r_{n,i} - 1 \right\}.$$

The underlying Bratteli diagram is properly ordered and the associated Bratteli-Vershik system $T_B: X_B \rightarrow X_B$ is topologically conjugate to $T: X \rightarrow X$. Note that $x_{\min} \in X_B$ corresponds to the point $y \in X$.

To sum up, a Bratteli diagram encodes a cutting-and-stacking process defining a system (see Figure 6).

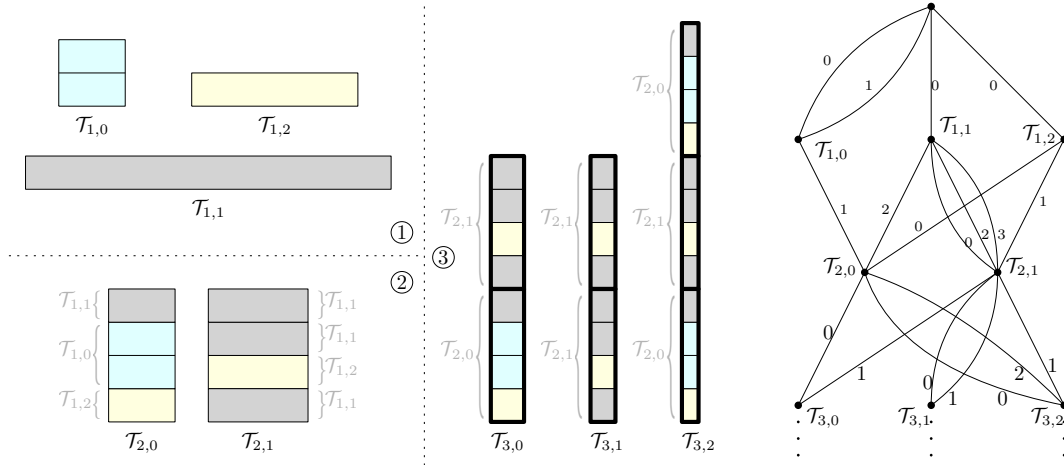


Figure 6: Example of towers $\mathcal{T}_{n,i}$, and the associated Bratteli diagram.

B.1.4 Classification up to strong orbit equivalence

Here we present a complete invariant of strong orbit equivalence, due to Giordano, Putnam and Skau.

Recall the incidence matrices

$$M_n := \left(m_{i,j}^{(n)} \right)_{\substack{0 \leq i \leq |V_{n+1}|-1 \\ 0 \leq j \leq |V_n|-1}}$$

given an enumeration of the vertices of each V_n . Then let us define the group $G(B)$ as the following inductive limit

$$G(B) := \lim \mathbb{Z}^{|V_0|} \xrightarrow{M_0} \mathbb{Z}^{|V_1|} \xrightarrow{M_1} \mathbb{Z}^{|V_2|} \xrightarrow{M_2} \dots$$

With the usual ordering on each $\mathbb{Z}^{|V_n|}$, $G(B)$ has a structure of ordered group, the unit order is chosen as the image of 1 in $\mathbb{Z} = \mathbb{Z}^{|V_0|}$. The ordered group $G(B)$ is called the **dimension group** of B . We refer the reader to [GPS95] for more details.

For instance, for the dyadic odometer, the incidence matrices are all (1×1) -matrices equal to (2), and the dimension group is $\mathbb{Z}[1/2]$.

Theorem B.1 (Giordano, Putnam, Skau [GPS95]). *Let S and T be two Cantor minimal homeomorphisms. The following assertions are equivalent:*

1. S and T are strongly orbit equivalent;
2. If B (resp. B') denotes a Bratteli diagram associated to S (resp. T), then the dimension groups $G(B)$ and $G(B')$ with distinguished order unit are order isomorphic.

B.2 Bratteli diagrams of odomutants

Let $X := \prod_{n \geq 0} \{0, \dots, q_n - 1\}$. Denoting by $\mathcal{P}(n)$ the partition whose atoms are the n -cylinders, with $\mathcal{P}(0) = (X)$, note that the sequence $(\mathcal{P}(n))_{n \geq 0}$ generates the infinite product topology on X and $\mathcal{P}(n+1)$ is composed of q_n towers of height h_n , denoted by

$$\mathcal{T}_{n+1,i} := \{B_{n+1,i}, T(B_{n+1,i}), \dots, T^{h_n-1}(B_{n+1,i})\}$$

where $B_{n+1,i} := [0, \dots, 0, i]_{n+1}$, for every $i \in \{0, \dots, q_n - 1\}$ (see Figure 2). The atoms of $\mathcal{T}_{n+1,i}$ are the cylinders of the form $[x_0, \dots, x_{n-1}, i]_{n+1}$ with $x_k \in \{0, \dots, q_k - 1\}$ for every $k \in \{0, \dots, n-1\}$.

Given $n \geq 1$ and $i \in \{0, \dots, q_n - 1\}$, the tower $\mathcal{T}_{n+1,i}$ visits the n -th towers with the following order:

$$\mathcal{T}_{n,(\sigma_i^{(n-1)})^{-1}(0)}, \mathcal{T}_{n,(\sigma_i^{(n-1)})^{-1}(1)}, \dots, \mathcal{T}_{n,(\sigma_i^{(n-1)})^{-1}(q_n-1)}.$$

According to Section B.1.3, we get the Bratteli diagram B of T illustrated in Figure 7.

The following map

$$\Psi: (x_n)_{n \geq 0} \in X \mapsto \left((v^{(0)}, v_{x_0}^{(1)}, 0), (v_{x_0}^{(1)}, v_{x_1}^{(2)}, \sigma_{x_1}^{(0)}(x_0)), (v_{x_1}^{(2)}, v_{x_2}^{(3)}, \sigma_{x_2}^{(1)}(x_1)), \dots \right) \in X_B$$

is a conjugation between T and the Bratteli-Vershik system T_B , it satisfies

$$\Psi(\psi^{-1}(X_\infty^+)) = X_B \setminus X_{B,\max}$$

$$\text{and } \Psi(\psi^{-1}(X_\infty^-)) = X_B \setminus X_{B,\min}.$$

In the case the permutations satisfy $\sigma_i^{(n)}(0) = 0$, $\sigma_i^{(n)}(q_n - 1) = q_n - 1$ for every $n \geq 0$, the Bratteli diagram is properly ordered and we have

$$X_{B,\max} = \{\Psi(x^+)\}$$

$$\text{and } X_{B,\min} = \{\Psi(x^-)\}.$$

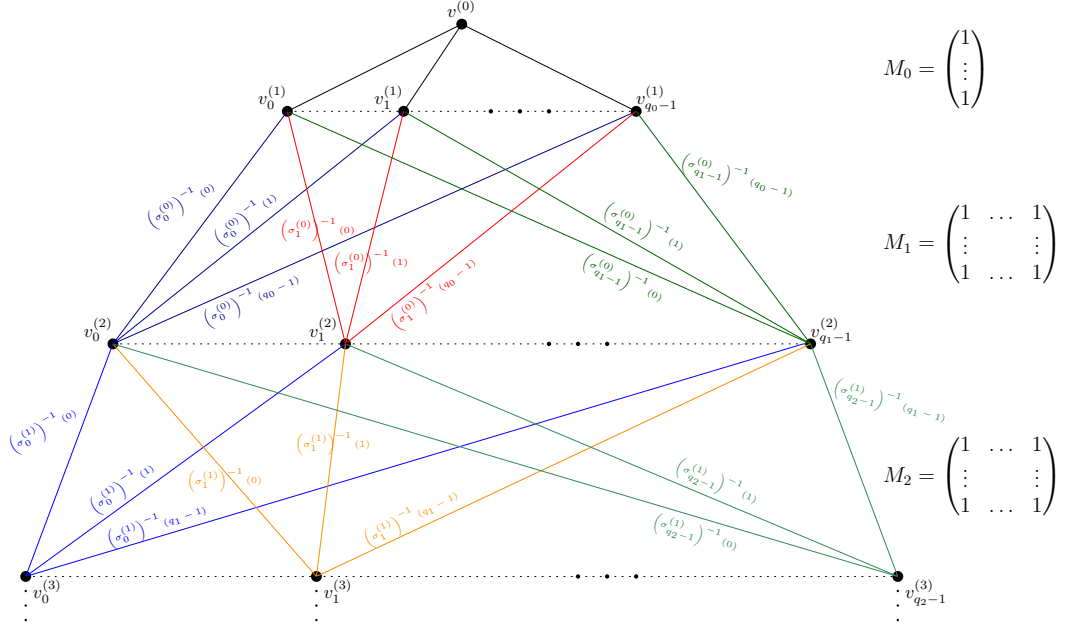


Figure 7: An ordered Bratteli diagram describing the odomutant built from the odometer on $\prod_{n \geq 0} \{0, \dots, q_n - 1\}$ and families of permutations $(\sigma_i^{(n)})_{0 \leq i < q_{n+1}}$ for $n \geq 0$.

B.3 Comparisons between Boyle and Handelmann's system and our odomutants.

As mentioned in the introduction, Boyle and Handelmann have shown the following result.

Theorem (Boyle, Handelmann [BH94]). *Let S be the dyadic odometer. Let α be either a positive real number or $+\infty$. Then there exists a Cantor minimal homeomorphism T such that:*

1. S and T are strongly orbit equivalent;
2. $h_{\text{top}}(T) = \alpha$.

In their proof, they build a Bratteli diagram B_{BH} (see Figure 8) similar to the diagram in Figure 7, the only difference is that for every $k \geq 1$, for every $v_i^{(k)} \in V_k$, $v_j^{(k+1)} \in V_{k+1}$, with $0 \leq i < q_{k-1}$ and $0 \leq j \leq q_k - 1$, there are n_k edges connecting these vertices. Then the ideas remain almost the same. Every vertex $v_j^{(k+1)} \in V_{k+1}$ provides a permutation $\sigma_j^{(k-2)}$ on the $n_k q_{k-1}$ edges of range $v_j^{(k+1)}$, satisfying

$$\sigma_j^{(k-2)}(0) = 0$$

$$\text{and } \sigma_j^{(k-2)}(n_k q_{k-1} - 1) = n_k q_{k-1} - 1,$$

so that the diagram is properly ordered and the associated Bratteli-Vershik system $T := T_{B_{\text{BH}}}$ can be extended to a homeomorphism on the Cantor set. The permutations are chosen in order to get $h_{\text{top}}(T) = \alpha$ (we refer the reader to their proof for more details, note that their proof in the case $\alpha = +\infty$ has been here entirely reformulated in our formalism).

It turns out that their Bratteli diagram is a diagram for an odomutant. Let us recall the following facts.

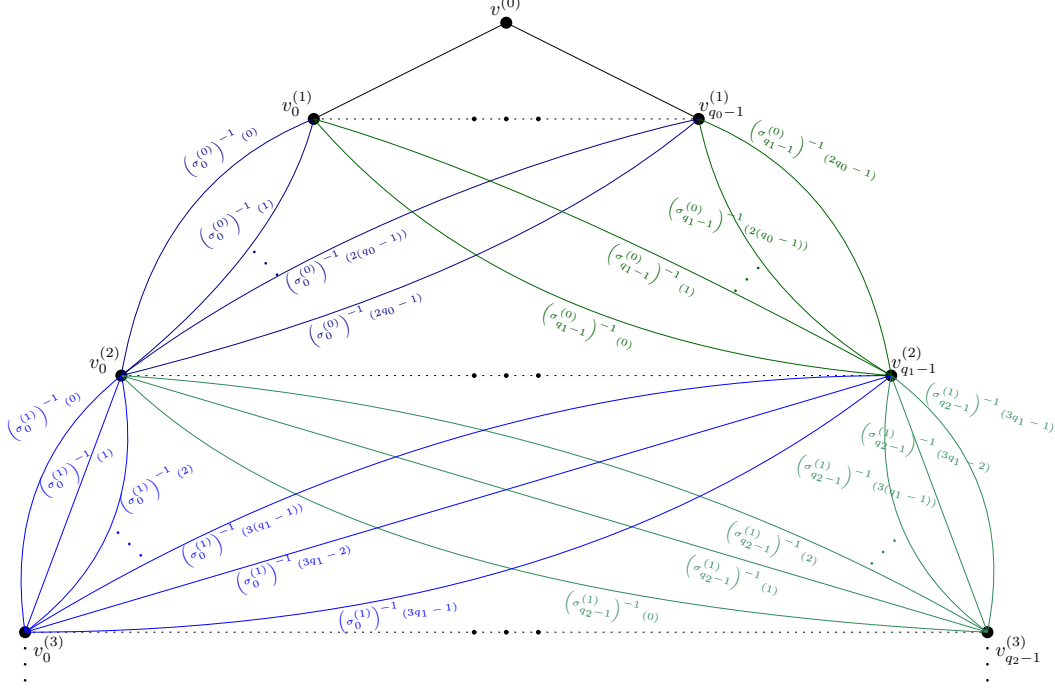


Figure 8: Bratteli diagram built by Boyle and Handelman in the proof of their Theorem 2.8 [BH94], with $n_1 = 2$, $n_2 = 3$.

- In a Bratteli diagram, for some fixed vertex $v_j^{(k+1)} \in V_{k+1}$, with $0 \leq j \leq q_k - 1$, the set of edges $r^{-1}(v_j^{(k+1)})$, with its linear ordering, encodes the stacking of subtowers of $\mathcal{T}_{k,0}, \dots, \mathcal{T}_{k,q_k-1}$ to build the tower $\mathcal{T}_{k+1,j}$ (see Figure 6).
- In the cutting-and-stacking construction of an odomutant described in Figure 2, every tower $\mathcal{T}_{k,i}$, with $0 \leq i < q_{k-1}$, is uniformly cut in q_{k+1} subtowers $(\mathcal{T}_{k,i}(\ell))_{0 \leq \ell < q_{k+1}}$ and we build every tower $\mathcal{T}_{k+1,j}$, with $0 \leq j < q_k$, by choosing only one subtower in each $\mathcal{T}_{k,0}, \dots, \mathcal{T}_{k,q_k-1}$ and stacking them.

The Bratteli diagram B_{BH} of Boyle and Handelman describes the following cutting-and-stacking construction: every tower $\mathcal{T}_{k,i}$ is uniformly cut in $n_k q_{k+1}$ subtowers $(\mathcal{T}_{k,i}(\ell))_{0 \leq \ell \leq n_k q_{k+1} - 1}$ and we build every tower $\mathcal{T}_{k+1,j}$ by choosing exactly n_k subtowers in each $\mathcal{T}_{k,0}, \dots, \mathcal{T}_{k,q_k-1}$ and stacking them.

As explained in Section 3.2, to understand why this is equivalent to the construction of an odomutant, it suffices to cut each k -th tower $\mathcal{T}_{k,i}$ in n_k (sub)towers $\mathcal{T}_{k,(i,0)}, \dots, \mathcal{T}_{k,(i,n_k-1)}$, in such a manner that for every $(k+1)$ -th tower $\mathcal{T}_{k+1,j}$, each tower $\mathcal{T}_{k,(i,m)}$ contains only one of the n_k subtowers $\mathcal{T}_{k,i}(\ell)$ which form $\mathcal{T}_{k+1,j}$. We then replace the former k -th towers $\mathcal{T}_{k,i}$, with $0 \leq i < q_{k-1}$, by the new ones $\mathcal{T}_{k,(i,m)}$, with $0 \leq i < q_{k-1}$ and $0 \leq m \leq n_k - 1$, and we recover the cutting-and-stacking process of an odomutant described above (with new integers n_k equal to 1). In other words, each vertex in V_k is split in n_k copies, and we get the Bratteli diagram B'_{BH} illustrated in Figure 9.

An infinite path of $X_{B_{\text{BH}}}$ can be uniquely written as

$$\left((v_{i_k}^{(k)}, v_{i_{k+1}}^{(k+1)}, (\sigma_{i_{k+1}}^{(k-1)})^{-1}(n_k i_k + m_k)) \right)_{k \geq 1}$$

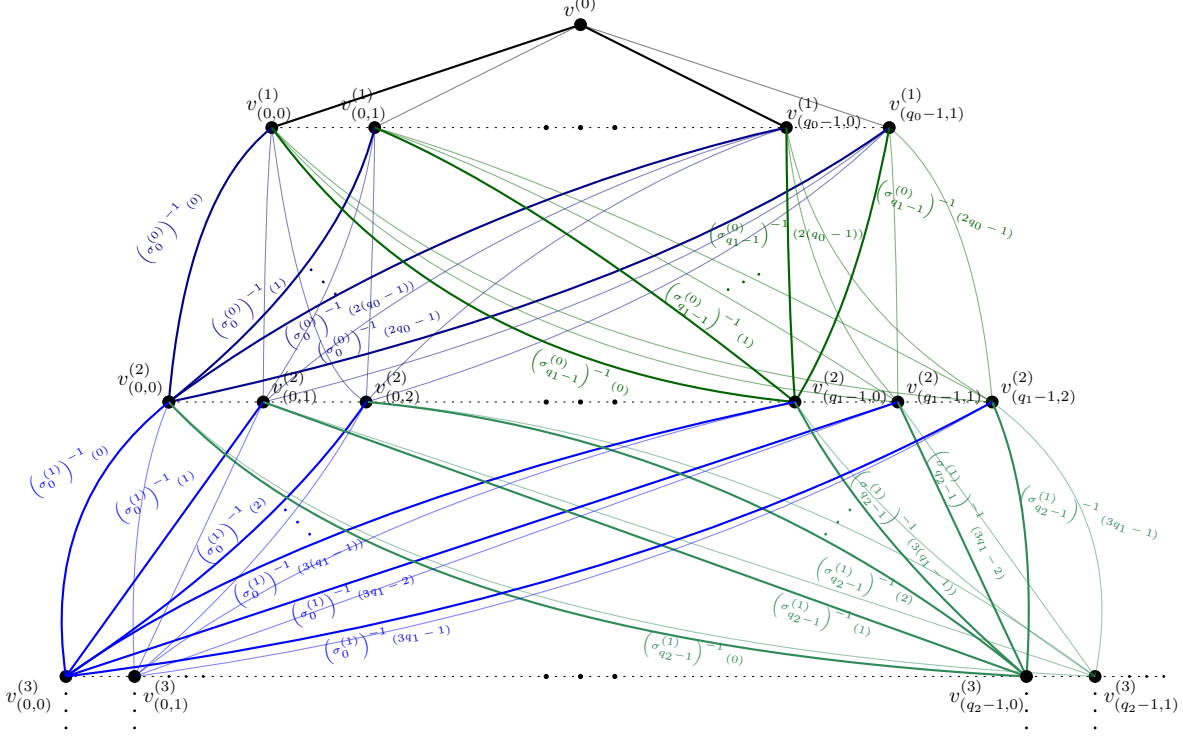


Figure 9: To get $B'_{\text{BH}} = (V', E')$ from the Bratteli diagram B_{BH} of Boyle and Handelmann (see Figure 8), we successively duplicate the vertices and the edges.

In B_{BH} , the integer n_1 is equal to 2 (from each vertex in V_1 to each one in V_2 , there are two edges), so each vertex $v_i^{(1)}$ (with $0 \leq i < q_0$) is split in two new vertices $v_{(i,0)}^{(1)}$ and $v_{(i,1)}^{(1)}$, each one being associated to one of the two edges in $r^{-1}(v_j^{(2)})$ (for every $0 \leq j < q_1$). For every $0 \leq m_1 \leq n_1 - 1$, there is only one edge between $v^{(0)}$ to $v_{i,m_1}^{(1)}$ (this edge can be considered as a copy of the edge from $v^{(0)}$ to the former vertex $v_i^{(1)}$).

The integer n_2 is equal to 3 (from each vertex in V_2 to each one in V_3 , there are three edges), so each vertex $v_j^{(2)}$ (with $0 \leq j < q_1$) is split in three new vertices $v_{(j,0)}^{(2)}$, $v_{(j,1)}^{(2)}$ and $v_{(j,2)}^{(2)}$, each one being associated to one of the three edges in $r^{-1}(v_k^{(3)})$ (for every $0 \leq k < q_2$). For every $0 \leq m_2 \leq n_2 - 1$, we define the edges of range $v_{j,m_2}^{(2)}$ as copies of the edges of range the former vertex $v_j^{(2)}$. A thicker edge corresponds to one copy. We do not indicate the rank of the other edges (the thinner ones) for clarity.

Then we apply the same algorithm to define the new vertices and edges in $E'_2, V'_3, E'_3, V'_4, \dots$

with $0 \leq i_k < q_{k-1}$ and $0 \leq m_k \leq n_k - 1$ (we omit the first edge $(v^{(0)}, v_{i_1}^{(1)}, 0)$). With the notations of Figure 9, the map

$$\begin{aligned} & \left((v_{i_k}^{(k)}, v_{i_{k+1}}^{(k+1)}), \left(\sigma_{i_{k+1}}^{(k-1)} \right)^{-1} (n_k i_k + m_k) \right)_{k \geq 1} \in X_{B_{\text{BH}}} \\ & \mapsto \left((v_{(i_k, m_k)}^{(k)}, v_{(i_{k+1}, m_{k+1})}^{(k+1)}), \left(\sigma_{i_{k+1}}^{(k-1)} \right)^{-1} (n_k i_k + m_k) \right)_{k \geq 1} \in X_{B'_{\text{BH}}} \end{aligned}$$

is a conjugation between the Bratteli-Vershik systems $T_{B_{\text{BH}}}$ and $T_{B'_{\text{BH}}}$.

B.4 Comparisons between Boyle and Handelmann's proof and our techniques.

Unlike Boyle and Handelmann, we prove the case $\alpha < +\infty$ of Theorem C with a cutting-and-stacking process where all new towers contain only one copy of each former tower. This is

naturally the construction encoded by an odomutant endowed with the sequence $(\mathcal{P}(\ell))_{\ell \geq 1}$ of partitions in ℓ -cylinders (see Figure 2). In order to get the case $\alpha = +\infty$, the main trick is to understand that a less restrictive cutting-and-stacking process, namely where every former tower may appear many times in the new ones, is encoded by an odomutant equipped with another sequence of partitions. Here the partitions are the ones associated to a description of this odomutant by multiple permutations, namely the partitions $\tilde{\mathcal{P}}(\ell)$ (see Definition 3.3). A first way to understand why this is relevant is to notice that with these partitions, we cannot distinguish between towers of the same step, as if they were the copies of the same former tower which appear in a new one (see Figure 3). Another remark is that Boyle and Handelmann use the partitions in cylinders in the Cantor space $X_{B_{BH}}$ on which their system $T_{B_{BH}}$ is defined. Therefore, if we want to reformulate their proof in our formalism and with the odomutant conjugate to $T_{B_{BH}}$, we have to consider the image of these partitions by the conjugation that we explicit above. It turns out that we get the partitions $\tilde{\mathcal{P}}(\ell)$.

To prove that the system $T_{B_{BH}}$ is strongly orbit equivalent to the dyadic odometer S , Boyle and Handelmann use the Giordano-Putnam-Skau theorem and the fact that the dimension group of $T_{B_{BH}}$ is $\mathbb{Z}[1/2]$. In our proof of Theorem C, the orbit equivalence is explicit and this enables us to directly show that the cocycles have at most one point of discontinuity. This also enables us to quantify the integrability of this orbit equivalence.

C Equivalence between definitions of loose Bernoullicity in the zero-entropy case

To our knowledge, justifications for the equivalence between two definitions of loose Bernoullicity in the zero-entropy case (see Theorem 2.8) is missing in the literature. Here we provide a proof. Let us first recall these definitions, that we already wrote in Section 2.4.

Definition C.1. Let $T \in \text{Aut}(X, \mu)$ and \mathcal{P} be a partition of X .

- (T, \mathcal{P}) is loosely Bernoulli, and we write T is **LB**, if for every $\varepsilon > 0$, for every sufficiently large integer N and for each $M > 0$, there exists a collection \mathcal{G} of "good" atoms in \mathcal{P}_{-M}^0 whose union has measure greater than or equal to $1 - \varepsilon$, and so that for each pair A, B of atoms in \mathcal{G} , the following holds: there is a probability measure $n_{A,B}$ on $\mathcal{P}^N \times \mathcal{P}^N$ satisfying
 - (I) $n_{A,B}(\{w\} \times \mathcal{P}^N) = \mu_A(\{[\mathcal{P}]_{1,N}(\cdot) = w\})$ for every $w \in \mathcal{P}^N$;
 - (II) $n_{A,B}(\mathcal{P}^N \times \{w'\}) = \mu_B(\{[\mathcal{P}]_{1,N}(\cdot) = w'\})$ for every $w' \in \mathcal{P}^N$;
 - (III) $n_{A,B}(\{(w, w') \in \mathcal{P}^N \times \mathcal{P}^N \mid f_N(w, w') > \varepsilon\}) < \varepsilon$.
- We say that (T, \mathcal{P}) is **LB₀** if for every $\varepsilon > 0$ and for every sufficiently large integer N , there exists a collection \mathcal{H} of "good" atoms in \mathcal{P}_1^N whose union has measure greater than or equal to $1 - \varepsilon$ and so that we have $f_N(w, w') \leq \varepsilon$ for every $w, w' \in [\mathcal{P}]_{1,N}(\mathcal{H})$.

Theorem C.2. Let $T \in \text{Aut}(X, \mu)$ and \mathcal{P} be a partition of X . If $h_\mu(T, \mathcal{P}) = 0$, then (T, \mathcal{P}) is **LB** if and only if it is **LB₀**.

This theorem relies on the following key lemma which crucially uses the assumption on the entropy. Note that, when considering a set \mathcal{Q} of subsets of X , for instance a set of atoms of a partition, $\mu(\mathcal{Q})$ will abusively denote the measure of $\bigcup_{A \in \mathcal{Q}} A$.

Lemma C.3. *Let $T \in \text{Aut}(X, \mu)$ and \mathcal{P} be a partition of X , such that $h_\mu(T, \mathcal{P}) = 0$. Let $\alpha > 0$ and N be a positive integer. Then there exists an integer $M_0 \geq 0$ such that the following holds for every $M \geq M_0$: there exists a collection $(\mathcal{Q}_C)_{C \in \mathcal{P}_1^N}$ of disjoint subsets of \mathcal{P}_{-M}^0 such that*

- *for every $C \in \mathcal{P}_1^N$, for every $A \in \mathcal{Q}_C$, we have $\mu_A(C) \geq 1 - \sqrt{\alpha}$;*
- *for every $C \in \mathcal{P}_1^N$, we have $\mu(\mathcal{Q}_C) \geq (1 - 2\sqrt{\alpha})\mu(C)$.*

Note that the second item implies $\mu\left(\bigcup_{C \in \mathcal{P}_1^N} \mathcal{Q}_C\right) \geq 1 - 2\sqrt{\alpha}$.

Proof of Lemma C.3. By [Dow11, Fact 2.3.12], the assumption $h_\mu(T, \mathcal{P}) = 0$ implies that \mathcal{P}_1^N is $\mathcal{P}_{-\infty}^0$ -mesurable, where

$$\mathcal{P}_{-\infty}^0 := \sigma(\mathcal{P}_{-M}^0, M \geq 0),$$

namely $\mathcal{P}_{-\infty}^0$ is the σ -algebra generated by the increasing sequence of algebras $(\sigma(\mathcal{P}_{-M}^0))_{M \geq 0}$. Then the following holds for every $C \in \mathcal{P}_1^N$: for every $\eta > 0$, there exists $B_C \in \bigcup_{M \geq 0} \sigma(\mathcal{P}_{-M}^0)$ such that $\mu(C \Delta B_C) \leq \eta$. Applied to $\eta = \alpha\mu(C)$, this fact provides an integer $M_0 \geq 0$ such that every atom $C \in \mathcal{P}_1^N$ is closed to some $B_C \in \sigma(\mathcal{P}_{-M_0}^0)$, namely $\mu(C \Delta B_C) \leq \alpha\mu(C)$. Let us fix an integer $M \geq M_0$, and notice that B_C is also in $\sigma(\mathcal{P}_{-M}^0)$.

For every $C \in \mathcal{P}_1^N$, let us set

$$\mathcal{Q}_C := \{A \in \mathcal{P}_{-M}^0 \mid A \subset B_C, \mu_A(C) > \min(1 - \sqrt{\alpha}, 1/2)\}.$$

Given two distinct atoms $C, C' \in \mathcal{P}_1^N$, the sets \mathcal{Q}_C and $\mathcal{Q}_{C'}$ are disjoint, otherwise we would have an atom $A \in \mathcal{P}_{-M}^0$ lying in \mathcal{Q}_C and $\mathcal{Q}_{C'}$ and such that the following occurs:

$$\mu(A) \geq \mu(A \cap C) + \mu(A \cap C') = (\mu_A(C) + \mu_A(C'))\mu(A) > \mu(A),$$

a contradiction.

Given $C \in \mathcal{P}_1^N$, it remains to prove $\mu(\mathcal{Q}_C) \geq (1 - 2\sqrt{\alpha})\mu(C)$. Let us write

$$\mathcal{Q}_C^- := \{A \in \mathcal{P}_{-M}^0 \mid A \subset B_C\} \setminus \mathcal{Q}_C.$$

On the one hand, we have

$$\begin{aligned} \mu(B_C \cap C) &= \sum_{A \in \mathcal{Q}_C} \mu(A \cap C) + \sum_{A \in \mathcal{Q}_C^-} \mu(A \cap C) \\ &\leq \mu(\mathcal{Q}_C) + (1 - \sqrt{\alpha})\mu(\mathcal{Q}_C^-) \\ &= \mu(\mathcal{Q}_C) + (1 - \sqrt{\alpha})(\mu(B_C) - \mu(\mathcal{Q}_C)) \\ &= (1 - \sqrt{\alpha})\mu(B_C) + \sqrt{\alpha}\mu(\mathcal{Q}_C) \\ &\leq (1 - \sqrt{\alpha})(1 + \alpha)\mu(C) + \sqrt{\alpha}\mu(\mathcal{Q}_C). \end{aligned}$$

where the last inequality comes from

$$\mu(B_C) \leq \mu(B_C \Delta C) + \mu(C) \leq (1 + \alpha)\mu(C)$$

On the other hand, we have

$$\mu(B_C \cap C) \geq \mu(C) - \mu(B_C \Delta C) \geq (1 - \alpha)\mu(C).$$

Combining all these inequalities, we get

$$\mu(\mathcal{Q}_C) \geq \frac{1}{\sqrt{\alpha}} (1 - \alpha - (1 - \sqrt{\alpha})(1 + \alpha)) \mu(C) = (1 - \sqrt{\alpha})^2 \mu(C) \geq (1 - 2\sqrt{\alpha})\mu(C),$$

as wanted. □

Proof of Theorem C.2. Assume that (T, \mathcal{P}) is LB. Let us fix $\varepsilon \in]0, 1[$ and a sufficiently large integer N as in the definition of LB. With $\alpha > 0$ small enough so that

$$(1 - \sqrt{\alpha})(1 - \sqrt{\alpha} - \varepsilon) \geq 1 - 2\varepsilon$$

$$\text{and } 1 - 2\sqrt{\alpha} \geq \varepsilon,$$

we apply Lemma C.3 to get M and $(\mathcal{Q}_C)_{C \in \mathcal{P}_1^N}$ as described in the statement. By definition of LB associated to the quantities ε , N and M , we get $\mathcal{G} \subset \mathcal{P}_{-M}^0$ covering at least $1 - \varepsilon$ of the space, and a family $(n_{A,B})_{A,B \in \mathcal{G}}$ of probabilities on $\mathcal{P}^N \times \mathcal{P}^N$ satisfying items (I), (II) and (III). Let us define

$$\mathcal{H} := \{C \in \mathcal{P}_1^N \mid \mathcal{G} \cap \mathcal{Q}_C \neq \emptyset\}.$$

We first have

$$\begin{aligned} \mu(\mathcal{H}) &= \sum_{C \in \mathcal{H}} \mu(C) \geq \sum_{C \in \mathcal{H}} \sum_{A \in \mathcal{Q}_C \cap \mathcal{G}} \mu(C \cap A) \geq (1 - \sqrt{\alpha}) \sum_{C \in \mathcal{H}} \sum_{A \in \mathcal{Q}_C \cap \mathcal{G}} \mu(A) \\ &= (1 - \sqrt{\alpha}) \mu \left(\mathcal{G} \cap \bigcup_{C \in \mathcal{P}_1^N} \mathcal{Q}_C \right) \\ &\geq (1 - \sqrt{\alpha})(1 - \sqrt{\alpha} - \varepsilon) \\ &\geq 1 - 2\varepsilon. \end{aligned}$$

Secondly, let us consider $C, C' \in \mathcal{H}$ and let us prove that $w := [\mathcal{P}]_{1,N}(C)$ and $w' := [\mathcal{P}]_{1,N}(C')$ are f_N -close. By definition, we can pick $A \in \mathcal{G} \cap \mathcal{Q}_C$ and $B \in \mathcal{G} \cap \mathcal{Q}_{C'}$, and using items (I) and (II) we have

$$\begin{aligned} n_{A,B}(\{w\} \times \mathcal{P}^N) &\geq \mu_A(C) \geq 1 - \sqrt{\alpha} \\ \text{and } n_{A,B}(\mathcal{P}^N \times \{w'\}) &\geq \mu_B(C') \geq 1 - \sqrt{\alpha}. \end{aligned}$$

This implies

$$n_{A,B}(\{(w, w')\}) \geq 1 - 2\sqrt{\alpha} \geq \varepsilon,$$

so $f_N(w, w') \leq \varepsilon$ by item (III). We have proved that (T, \mathcal{P}) satisfies LB_0 for 2ε .

Let us now assume that (T, \mathcal{P}) is LB_0 , we fix $\varepsilon > 0$, a sufficiently large integer $N > 0$ and an associated $\mathcal{H} \subset \mathcal{P}_1^N$ as in the definition of LB_0 . With $\alpha > 0$ small enough so that

$$(1 - \sqrt{\alpha})^2 \geq 1 - \varepsilon$$

$$\text{and } (1 - 2\sqrt{\alpha})(1 - \varepsilon) \geq 1 - 2\varepsilon,$$

we apply Lemma C.3 to get M_0 and for every $M \geq M_0$, an associated collection $(\mathcal{Q}_C)_{C \in \mathcal{P}_1^N}$ as described in the statement. Let us fix $M \geq M_0$ and let us consider

$$\mathcal{G} := \bigcup_{C \in \mathcal{H}} \mathcal{Q}_C$$

and for every $A, B \in \mathcal{G}$, the probability $n_{A,B}$ on $\mathcal{P}^N \times \mathcal{P}^N$ defined by

$$n_{A,B}(\{(w, w')\}) = \mu_A(\{[\mathcal{P}]_{1,N}(\cdot) = w\}) \mu_B(\{[\mathcal{P}]_{1,N}(\cdot) = w'\}),$$

they automatically satisfy items (I) and (II). Given $C, C' \in \mathcal{H}$, $A \in \mathcal{Q}_C$ and $B \in \mathcal{Q}_{C'}$, and $w := [\mathcal{P}]_{1,N}(C)$ and $w' := [\mathcal{P}]_{1,N}(C')$, we have

$$n_{A,B}(\{(w, w')\}) \geq \mu_A(C) \mu_B(C') \geq (1 - \sqrt{\alpha})^2 \geq 1 - \varepsilon,$$

and since $f_N(w, w') \leq \varepsilon$, we get item (III). Finally, we have

$$\mu(\mathcal{G}) = \sum_{C \in \mathcal{H}} \mu(\mathcal{Q}_C) \geq (1 - 2\sqrt{\alpha}) \sum_{C \in \mathcal{H}} \mu(C) = (1 - 2\sqrt{\alpha})\mu(\mathcal{H}) \geq (1 - 2\sqrt{\alpha})(1 - \varepsilon) \geq 1 - 2\varepsilon.$$

We have proved that (T, \mathcal{P}) satisfies LB for 2ε , N large enough and $M \geq M_0$. By [Fel76, Corollary 2], we can replace "for each $M > 0$ " by "for every sufficiently large $M > 0$ " in the definition of LB, so we are done. \square

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